Chapter 7

Interpolation in Model Theory

Jouko Väänänen

University of Helsinki, Finland

─── **Abstract** ───

We bring an abstract model theory perspective to interpolation. We ask, what is the role of interpolation in the study of extensions of first order logic, such as infinitary logics, generalized quantifiers and higher order logics? The abstract model theory approach reveals the basic connections between various interpolation properties in isolation, on their own, as well as with respect to other model theoretic properties, such as compactness.

Contents

1 Introduction

The Craig Interpolation Theorem ([12]) was originally proved by proof theoretic methods. Even today proof theory or methods such as tableaux methods derived from proof theory are the most popular road to interpolation. The model theoretic version of interpolation is the Robinson property ([48]), which can be readily proved with purely model theoretic methods. The statement of the Robinson property resembles amalgamation which is at the heart of modern model theory. Neither interpolation nor the Robinson property seem to be central in current model theory but the idea of amalgamation is certainly central (see e.g.

[29]). Moreover, the model theoretic proof of the Robinson property is in principle based on the use of one form or another of saturated models and saturated models are certainly important in model theory.

In this chapter we take the axiomatic (or abstract) approach to model theory. We investigate which model theoretic properties of first order logic, which we denote in this chapter by $L_{\omega\omega}$, are behind interpolation and the Robinson property. For this to make sense we recall the concept of an abstract logic as well as a variety of extensions of first order logic, some with interpolation and some without. This indicates what it is in first order (or some other) logic that allows us to derive the interpolation theorem.

The topic of interpolation and its weaker forms in model theoretic languages is a vast topic and we will only touch upon some basic results in this chapter. The topic is so extensive because there are numerous extensions of first order logic as well as numerous weaker forms of interpolation. This leads to a huge potential for implications and counter-examples. We have chosen to include in this chapter only what we think as illustrative of the situation. A good reference to a more complete picture is still today [2].

Section 2 introduces the basic concepts of abstract model theory and the basic general results about compactness, interpolation, the Robinson property and the Beth Definability Theorem ([7]). We prove some basic relationships between these properties. The focus is on how much we have to assume of the logics for the relationships to hold. Section 3 concentrates on generalized quantifiers, wherein we present the basic facts. The amount of material here is huge but we focus on what we think is illuminating. Section 4 presents the basic facts about interpolation in infinitary logics ending with Shelah's relatively new logic L^1_κ, which has an interpolation property ([53]). Finally in Section 5 we discuss the matter of interpolation in higher order logics, including logics without negation such as existential second order logic and dependence logic ([55]).

We consider both single-sorted and many-sorted logic, as the difference between the two is relevant in the study of interpolation. Everything is many-sorted unless otherwise specified. As is well-known, many-sorted logic can be reduced to single-sorted logic, but this reduction is not so perfect that many-sorted vocabularies would be rendered useless.

The reader is assumed to know basic first order model theory. Otherwise we try to be self-contained. For details on generalized quantifiers, infinitary languages and abstract model theory, a good source is [2]. For unexplained concepts of set theory we refer to [31]. For basics of model theory we refer to [29].

2   The Abstract Setting

By abstract model theory we mean here the study of extensions of first order logic, such as logics with generalized quantifiers, infinitary logics and higher order logics. Interpolation by and large fails in such logics but there are notable exceptions, such as $L_{\omega_1\omega}$ (see Section 4), and in some cases interpolation can be established in a relative sense meaning that the

interpolant is found, not in the logic under consideration, but in a hopefully not too much bigger logic.

In its most general form, an *abstract logic*, semantically conceived, is just a triple $L = (S, F, \models)$ where S and F are classes and $\models$ is a subclass of $S \times F$. Elements of S are called the *structures* of L, elements of F are called the *sentences* of L, and the relation $\models$ is called the satisfaction relation of L. Note that we do not assume anything about the "structures" that constitute S. They need not be structures in the sense of ordinary first order model theory. They can also be Kripke models or valuations of propositional logic or whatnot. In this generality the concept of an abstract logic covers first order logic and its extensions by generalized quantifiers and infinitary operations, but also modal logic, propositional logic, topological logic [21], etc. Likewise, the elements of F, i.e. the "sentences" of L, need not be sentences in any usual sense. Normally sentences are identified with finite strings of symbols but sentences of infinitary logics, such as $L_{\omega_1 \omega}$ are best thought of as sets (or perhaps trees). Finally, the satisfaction relation $\models$ does not have to have any inductive definition tying together S and F. All we assume is that it is a subclass of $S \times F$, which means that it is a class in the sense of set theory[1] i.e. it is a definable predicate of set-theory, perhaps from some parameters.

On this level of generality it is, of course, hardly possible to prove any deep results about abstract logics. But if some simple results could be proved, they would automatically apply to first order logic, propositional logic, modal logic, and so on, and would manifest similarity or a "family resemblance" of the logics from this point of view, while the logics mentioned live otherwise lightyears from each other. Indeed, as we shall see, some very basic facts about interpolation and model theoretic properties around it can be proved even in our very general setup. Whether the results that can be proved in this generality are interesting or not, is a reasonable question. Be that as it may, it is however notable that we can perfectly formulate many fundamental model theoretic *concepts* on this level of generality, even if no deep results can be proved.

The basic concepts of abstract logics are the following: A structure $\mathfrak{M} \in S$ is a *model of* $\varphi \in F$ if $\mathfrak{M} \models \varphi$, and a *model of* $\Sigma \subseteq F$, in symbols $\mathfrak{M} \models \Sigma$, if $\mathfrak{M}$ is a model of each $\varphi \in \Sigma$. The sentence $\varphi \in F$, or a set $\Sigma \subseteq F$, *is consistent* if it has a model. Whenever $\varphi, \psi \in F$, we write $\varphi \models \psi$, if for every model $\mathfrak{M} \in S$, $\mathfrak{M} \models \varphi$ implies $\mathfrak{M} \models \psi$, and $\varphi \equiv \psi$ if $\varphi \models \psi$ and $\psi \models \varphi$. Whenever $\Sigma \cup \{\psi\} \subseteq F$, we write $\Sigma \models \psi$, if $\mathfrak{M} \models \Sigma$ implies $\mathfrak{M} \models \psi$ for all $\mathfrak{M} \in S$. Two models $\mathfrak{M}, \mathfrak{M}' \in S$ are *L-equivalent*, in symbols $\mathfrak{M} \equiv_L \mathfrak{M}'$, if they are models of the same sentences of F. A set $\Sigma \subseteq F$ is called *complete* if for some $\mathfrak{M} \in S$ we have $\Sigma = \{\varphi \in F : \mathfrak{M} \models \varphi\}$. Any consistent set $\Sigma \subseteq F$ can be extended to a complete consistent Σ' by taking a model $\mathfrak{M}$ of Σ and letting $\Sigma' = \{\varphi \in F : \mathfrak{M} \models \varphi\}$.

Already this very general approach imposes an equivalence relation $\equiv_L$ on S. Different abstract logics may impose different equivalence relations on the same class S. Intuitively

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[1] Nothing essential would change if we worked in class theory instead of set theory.

speaking a stronger logic imposes a finer equivalence relation, if S is fixed. Respectively, the structures in S impose relationships between sentences in F, e.g. the relation $\varphi \models \psi$.

We can identify a sentence with the class of its models. Likewise, we can identify a structure with the class of sentences it is a model of. Such identifications would lead to further abstraction without necessarily being helpful. But this demonstrates a certain duality between structures and sentences, well-known and much studied in the development of mathematical logic. Of course, these basic relationships between structures and sentences depend heavily on what is our (abstract) logic L.

▶ **Definition 1.** *We say that an abstract logic L satisfies the* Compactness Theorem *or is* compact*, if for every set $\Sigma \subseteq F$ the following holds: If every finite $\Gamma \subseteq \Sigma$ is consistent, then Σ itself is consistent. If this holds for all countable Σ, we say that L satisfies the* Countable Compactness Theorem *or is* countably compact*.

To formulate interpolation we introduce the *vocabulary* (class) function τ which has $S \cup F$ as its domain. The values of τ are arbitrary sets but intuitively $\tau(\varphi)$ is the set of relation, constant, proposition, function, etc -symbols occurring in the sentence φ. Intuitively, $\tau(\mathfrak{M})$ is the set of relation, constant, proposition, function, etc -symbols that are interpreted in $\mathfrak{M}$. We assume always that $\mathfrak{M} \models \varphi$ implies $\tau(\varphi) \subseteq \tau(\mathfrak{M})$. If $\Sigma \subseteq F$, we write $\tau(\Sigma)$ for $\bigcup_{\varphi \in \Sigma} \tau(\varphi)$. We could go ahead and formulate natural axioms that τ should satisfy in order to correspond to our intuition about the vocabulary of a sentence or of a structure. However, we skip that here, as it is not relevant for us now. We just call the quadruple $(S, F, \models, \tau)$ an abstract logic. For such a quadruple we can now define the concept of interpolation:

▶ **Definition 2.** *The abstract logic $L = (S, F, \models, \tau)$ has the* Interpolation property *if for every $\varphi, \psi \in F$, the relation $\varphi \models \psi$ implies the existence of $\theta \in F$ such that*

1. $\varphi \models \theta$
2. $\theta \models \psi$
3. $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$.

In the sense of this definition, first order logic, the infinitary logic $L_{\omega_1 \omega}$ ([32], see section 4 below), Shelah's L^1_κ ([53], see section 4 below), single-sorted second order logic (see section 5), propositional logic, modal logic and topological logic ([21]) all have the interpolation property. Still, considering how many extensions of first order logic have been introduced, see e.g. [2], it is a little surprising how few of them have the interpolation property. This circumstance has led to a study of weaker forms of interpolation as well as of the possible reasons why many logics fail to satisfy interpolation.

For the history of the birth of the Interpolation property of first order logic we refer to [15].

Assuming the Compactness Theorem, there is an alternative formulation of interpolation:

▶ **Definition 3.** *The abstract logic $L = (S, F, \models, \tau)$ has the* Robinson (Consistency) property *([48]) if for every complete $\Sigma_0 \subseteq F$, every consistent $\Sigma_1 \subseteq F$ extending Σ_0, and every consistent $\Sigma_2 \subseteq F$ extending Σ_0, such that $\tau(\Sigma_1) \cap \tau(\Sigma_2) = \tau(\Sigma_0)$, the set $\Sigma_1 \cup \Sigma_2$ is consistent.*

The model theoretic proof of the Robinson property for first order logic is based on the following simple idea: Take an infinite saturated model $\mathfrak{M}_1$ of Σ_1 and a saturated model $\mathfrak{M}_2$ of Σ_2 such that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have the same cardinality. Now $\mathfrak{M}_1 \restriction \tau(\Sigma_0)$ and $\mathfrak{M}_2 \restriction \tau(\Sigma_0)$ are saturated elementary equivalent (since Σ_0 is complete) models of the same cardinality. Hence they are isomorphic. The isomorphism can be used to transfer the interpretations of symbols in $\tau(\Sigma_2) \setminus \tau(\Sigma_0)$ from $\mathfrak{M}_2$ to $\mathfrak{M}_1$. This yields an expansion $\mathfrak{N}$ of $\mathfrak{M}_1$ such that $\mathfrak{N} \restriction \tau(\Sigma_1) = \mathfrak{M}_1$ and $\mathfrak{N} \restriction \tau(\Sigma_2) \cong \mathfrak{M}_2$. Thus $\mathfrak{N}$ is a model of $\Sigma_1 \cup \Sigma_2$. It should be noted that this proof is an overkill but worth knowing because it is probably the quickest proof. Indeed, saturated models are only known to exist if we assume some amount of the Generalized Continuum Hypothesis or alternatively the existence of strongly inaccessible cardinals, neither of which is really needed for the result. The use of saturated models can be avoided by more refined methods, for example special models ([11]) or recursively saturated models ([4]), which always exist. Alternatively, we present below in Theorem 17 a gentler proof due to Lindström. We will also prove below in Theorem 5 the Robinson property from compactness and interpolation in a very general setting.

To connect the Interpolation property and the Robinson property we need to assume a little bit about abstract logics. To this end we define:

▶ **Definition 4.** **1.** *The abstract logic $L = (S, F, \models, \tau)$ is* closed under conjunction *if for every $\varphi, \psi \in F$ there is $\theta \in F$, denoted $\varphi \wedge \psi$, such that $\tau(\theta) = \tau(\varphi) \cup \tau(\psi)$ and for all $\mathfrak{M} \in S$:*

$$\mathfrak{M} \models \varphi \wedge \psi \iff \mathfrak{M} \models \varphi \text{ and } \mathfrak{M} \models \psi.$$

2. *L is* closed under negation *if for every $\varphi \in F$ there is $\theta \in F$, denoted $\neg\varphi$, such that $\tau(\theta) = \tau(\varphi)$ and for all $\mathfrak{M} \in S$:*

$$\mathfrak{M} \models \neg\varphi \iff \mathfrak{M} \not\models \varphi.$$

3. *If L is closed under both conjunction and negation, we denote $\neg(\neg\varphi \wedge \neg\psi)$ by $\varphi \vee \psi$ and $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$ by $\varphi \leftrightarrow \psi$.*

▶ **Theorem 5.** *If L is closed under negation and conjunction, and satisfies the Compactness Theorem, then the following conditions are equivalent:*
1. *L has the Interpolation property.*
2. *L has the Robinson property.*

Rather than proving this, we formulate and prove a stronger version. Since in abstract model theory there are numerous examples of the failure of the Interpolation property as well as of the Robinson property, it makes sense to introduce relative versions of both. To this end we need the concept of a *sublogic*:

▶ **Definition 6.** **1.** *A logic $L = (S, F, \models, \tau)$ is a sublogic of $L' = (S', F', \models', \tau')$, in symbols $L \leq L'$, if $S = S'$ and for every $\varphi \in F$ there is $\varphi' \in F'$ such that $\varphi \equiv \varphi'$ i.e. for all $\mathfrak{M} \in S$, $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M} \models' \psi$, and in addition, $\tau(\varphi) = \tau'(\varphi')$. We usually identify φ and φ' and thereby $F \subseteq F'$.*

2. *Logics L and L' are* equivalent, *$L \equiv L'$, if $L \leq L'$ and $L' \leq L$.*

Note that here both logics in the above definition have the same class S of structures. The more general case of different classes of structures can be dealt with by means of the so-called Chu-transform, see [17]. Note that a sublogic of a (countably) compact logic is (countably) compact.

▶ **Definition 7.** *Suppose $L = (S, F, \models, \tau)$ and $L' = (S', F', \models', \tau')$ are abstract logics such that $L \leq L'$.*

1. $\mathrm{Craig}(L, L')$ *holds if for every $\varphi, \psi \in F$, the relation $\varphi \models \psi$ implies the existence of $\theta \in F'$ such that for all $\mathfrak{M} \in S$, $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models' \theta$, $\mathfrak{M} \models' \theta \Rightarrow \mathfrak{M} \models \psi$, and $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$. By $\mathrm{Craig}(L)$ we mean $\mathrm{Craig}(L, L)$.*

2. $\mathrm{Rob}(L, L')$ *holds if for every complete $\Sigma_0 \subseteq F'$, every $\Sigma_1 \subseteq F$, such that $\Sigma_0 \cup \Sigma_1$ is consistent, and every $\Sigma_2 \subseteq F$, such that $\Sigma_0 \cup \Sigma_2$ is consistent, if $\tau(\Sigma_0 \cup \Sigma_1) \cap \tau(\Sigma_0 \cup \Sigma_2) = \tau(\Sigma_0)$, then the set $(\Sigma_0 \cap F) \cup \Sigma_1 \cup \Sigma_2$ is consistent. By $\mathrm{Rob}(L)$ we mean $\mathrm{Rob}(L, L)$.*

▶ **Theorem 8.** *If $L \leq L'$ are closed under negation and conjunction, and L' satisfies the Compactness Theorem, then the following are equivalent:*

1. $\mathrm{Craig}(L, L')$.

2. $\mathrm{Rob}(L, L')$.

Proof. We follow the standard proof, as e.g. in [2, Chapter II, 7.1.5.].

(1) implies (2): Suppose a complete $\Sigma_0 \subseteq F'$ is given. Suppose also $\Sigma_1 \subseteq F$, such that $\Sigma_0 \cup \Sigma_1$ is (wlog) complete, and $\Sigma_2 \subseteq F$, such that $\Sigma_0 \cup \Sigma_2$ is (wlog) complete, are given, and furthermore $\tau(\Sigma_0 \cup \Sigma_1) \cap \tau(\Sigma_0 \cup \Sigma_2) = \tau(\Sigma_0)$. Finally, suppose the set $(\Sigma_0 \cap F) \cup \Sigma_1 \cup \Sigma_2$ is inconsistent. By the Compactness Theorem of L' there are some $\{\eta_0, \ldots, \eta_k\} \subseteq \Sigma_0 \cap F$, $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Sigma_1$ and $\{\psi_1, \ldots, \psi_m\} \subseteq \Sigma_2$ such that $\{\eta_0, \ldots, \eta_k\} \cup \{\varphi_1, \ldots, \varphi_n\} \cup \{\psi_1, \ldots, \psi_m\}$ is inconsistent. Since $\Sigma_0 \cup \Sigma_1$ and $\Sigma_0 \cup \Sigma_2$ are individually both consistent, we have $n > 0$ and $m > 0$. Let $\eta = \eta_1 \wedge \ldots \wedge \eta_n$, $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_n$ and $\psi = \neg(\psi_1 \wedge \ldots \wedge \psi_m)$. Thus $\eta \wedge \varphi \models \psi$. Let $\theta \in F$ such that $\eta \wedge \varphi \models \theta$, $\theta \models \psi$ and $\tau(\theta) \subseteq \tau(\eta \wedge \varphi) \cap \tau(\psi)$. Since $\Sigma_0 \cup \Sigma_1$ is complete, $\theta \in \Sigma_0 \cup \Sigma_1$. Since $\Sigma_0 \cup \Sigma_2$ is complete and consistent, $\neg\theta \in \Sigma_0 \cup \Sigma_2$. Since Σ_0 is complete, $\{\theta, \neg\theta\} \subseteq \Sigma_0$, contrary to the consistency of Σ_0.

(2) implies (1): Suppose $\varphi \models \psi$, where $\varphi, \psi \in F$ and $\tau_0 = \tau(\varphi) \cap \tau(\psi)$. Let Σ_0 be the set of $\theta \in F'$ such that $\varphi \models \theta$ and $\tau(\theta) \subseteq \tau_0$. We show now that $\Sigma_0 \models \psi$. Otherwise there is a model $\mathfrak{M}$ of $\Sigma_0 \cup \{\neg\psi\}$. Let Σ_0^* be the set of $\theta \in F'$ such that $\mathfrak{M} \models \theta$ and $\tau(\theta) \subseteq \tau_0$. Note that Σ_0^* is complete. The set $\Sigma_0^* \cup \{\psi\}$ is consistent for otherwise the Compactness Theorem gives $\{\theta_1, \ldots, \theta_k\} \subseteq \Sigma_0^*$ such that $\psi \models \neg(\theta_1 \wedge \ldots \wedge \theta_k)$ implying $\neg(\theta_1 \wedge \ldots \wedge \theta_k) \in \Sigma_0$, a contradiction. Now both $\Sigma_0^* \cup \{\neg\psi\}$ and $\Sigma_0^* \cup \{\psi\}$ are consistent. By (2), $(\Sigma_0^* \cap F) \cup \{\varphi\} \cup \{\neg\psi\}$

is consistent, a contradiction. Having now proved $\Sigma_0 \models \psi$ we use again the Compactness Theorem. We obtain $\{\theta_1, \ldots, \theta_k\} \subseteq \Sigma_0$ such that $\{\theta_1, \ldots, \theta_k\} \models \psi$. Letting $\theta = \theta_1 \wedge \ldots \wedge \theta_k$ yields $\theta \in F'$, $\varphi \models \theta$, $\theta \models \psi$ and $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$. ◀

In the above theorem it is enough to assume that L' is countably compact, provided that L' satisfies the condition that there are only countably many sentences with a given countable vocabulary and the property $\mathrm{Rob}(L, L')$ is formulated for countable vocabularies only.

▶ **Definition 9.** *An abstract logic* $(S, F, \models, \tau)$ *is* classical, *if:*
1. $\tau(\varphi)$ *consists, for all* $\varphi \in F$*, of relation, constant and function symbols.*
2. S *is a class of structures* $\mathfrak{M}$ *in the sense of first order logic and* $\tau(\mathfrak{M})$ *has its usual meaning as the vocabulary of the structure* $\mathfrak{M}$*.*
3. *If* $\mathfrak{M} \in S$ *and* $a_1, \ldots, a_n \in M$*, we assume that the structure* $(\mathfrak{M}, a_1, \ldots, a_n)$*, obtained from* $\mathfrak{M}$ *by distinguishing the elements* $a_1, \ldots, a_n$ *as interpretations of new constant symbols* $c_1, \ldots, c_n$ *is also in* S*.*
4. *If* σ *is a subvocabulary of* $\tau(\mathfrak{M})$*, then the* reduct $\mathfrak{M} {\restriction} \sigma$ *is in* S*.*
5. *If* $\mathfrak{M} \in S$ *and* $\varphi \in F$*, then* $\mathfrak{M} \models \varphi$ *iff* $\mathfrak{M} {\restriction} \tau(\varphi) \models \varphi$*.*
6. L *satisfies renaming[2] in the sense of Definition 1.1.1 of [2].*
7. *A formula[3]* $\varphi(x_1, \ldots, x_n)$ *of* L *is a sentence* $\varphi(c_1, \ldots, c_n)$*, where* $c_1, \ldots, c_n$ *are new constant symbols.*

A classical logic L *is* atomic *if for every relation symbol* $R(x_1, \ldots, x_n)$ *and constant symbols* $c_1, \ldots, c_n$ *there is a sentence* $\varphi(c_1, \ldots, c_n) \in F$*, such that* $\tau(\varphi(c_1, \ldots, c_n)) = \{R, c_1, \ldots, c_n\}$ *and for all* $\mathfrak{M} \in S$ *with* $\tau(\varphi) \subseteq \tau(\mathfrak{M})$*, and all* $a_1, \ldots, a_n \in M$*.*

$$(\mathfrak{M}, a_1, \ldots, a_n) \models \varphi(c_1, \ldots, c_n) \iff (a_1, \ldots, a_n) \in R^{\mathfrak{M}}.$$

We write $\varphi(c_1, \ldots, c_n)$ *as* $R(c_1, \ldots, c_n)$*.*

Apart from ordinary first order logic and its extension by generalized quantifiers, infinitary logical operations, and higher order quantifiers, first order logic with finite models is a classical atomic abstract logic. Also first order logic with ordered models, i.e. models with a distinguished linear order $<$, which is an element of every vocabulary considered, and does not disappear in a reduct, is a classical atomic abstract logic. The two-variable fragment of first order logic, where vocabularies consist only of binary and unary predicates symbols, constant symbols and contain no function symbols, and only two variables are used overall, is a classical atomic abstract logic.

A stronger but still natural and prevalent property of logics is *regularity* in the sense of [2, Chapter II].

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[2] Essentially, renaming says that we can change symbols in a formula and truth is preserved, if we make the respective changes in structures.
[3] We use constant symbols to play the role of variables.

For classical logics it makes sense to talk about the cardinality of the models. In particular, we can formulate the important *Löwenheim property*: Every sentence with a model has a countable model.

An EC(L)-class ("E" for "elementary") is a subclass K of S such that for some $\varphi \in F$, $\mathfrak{M} \in K$ if and only if $\mathfrak{M} \models \varphi$. We then write $\tau(K) = \tau(\varphi)$. A PC(L)-class ("P" for "projective") is a subclass K of S such that $\tau(\mathfrak{M})$ is a fixed τ for $\mathfrak{M} \in K$, and for some $\varphi \in F$, $\mathfrak{M} \in K$ if and only if $\mathfrak{N} \models \varphi$ for some $\mathfrak{N}$ such that $\mathfrak{N} \restriction \tau = \mathfrak{M}$[4]. We then write $\tau(K) = \tau$. Here $\tau(\varphi)$ may have more sorts than τ[5]. If no new sort occur in $\tau(\varphi)$, then PC($L_{\omega\omega}$)-definability coincides with existential second order definability. We may consider the family of all PC(L)-classes an abstract logic in the obvious sense.

Craig [13] showed that his interpolation theorem has an equivalent formulation as a separation property:

▶ **Definition 10.** *The* separation property Sep(L, L') *holds if any two disjoint PC(L)-classes K_0 and K_1 with the same vocabulary τ can be separated by an EC(L')-class K, i.e. $K_0 \subseteq K$ and $K \cap K_1 = \emptyset$ with $\tau(K) = \tau$. A logic L satisfies the* separation property *if it satisfies* Sep(L, L). *Two logics L and L' satisfy the* Souslin-Kleene interpolation property, SK(L, L'), *if every PC(L)-class K whose complement in the class $\{\mathfrak{M} \in S : \tau(\mathfrak{M}) = \tau(K)\}$ is also a PC(L)-class, is an EC(L')-class.*

Naturally Sep(L, L') implies SK(L, L'), but not conversely ([30]). Note that Sep(L, L') implies $L \leq L'$, if L is closed under negation. The family of all PC(L)-classes can be construed as a logic itself, as is carefully explained in ([42]). This logic, denoted $\Delta(L)$, has similar regularity properties as L. For example, it is classical, atomic and closed under conjunction and negation if L is. It is the smallest extension of a regular abstract logic L to a regular abstract logic with the Souslin-Kleene Interpolation property ([42]). In fact, the Souslin-Kleene Interpolation property is also called the Δ-interpolation property ([9]). If we limit ourselves to PC(L)-classes that do not add new sorts (i.e. the defining formula does not have new sorts, only new predicates, functions and constants) we obtain a more restrictive extension $\Delta_1^1(L)$ which behaves very much in the same way as $\Delta(L)$. Of course, $\Delta_1^1(L) \subseteq \Delta(L)$.

▶ **Example 11.** The class of models (M, E), where $E \subseteq M \times M$ is an equivalence relation with uncountably many uncountable classes, is definable in $\Delta(L(Q_1))$ but not in $L(Q_1)$ (see Theorem 19, especially its proof). Hence $\Delta(L(Q_1))$ is a proper extension of $L(Q_1)$.

We prove the following simple result, generalizing a result from [13], in some detail because we assume in a sense the minimal amount of the logics, so it is not so clear that the classical proof works. It is perhaps interesting to see where the different assumptions are used.

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[4] For example, the class K of infinite models of the empty vocabulary is a PC($L_{\omega\omega}$)-class as we can let φ be the first order sentence of vocabulary $\{<\}$ which says that $<$ is a linear order without last element.

[5] This will be relevant when we discuss second order logic.

▶ **Proposition 12.** *Assume L and L' are classical and satisfy $S = S'$. Suppose also that L is closed under negation. The following conditions are equivalent:*
1. $\text{Craig}(L, L')$.
2. $\text{Sep}(L, L')$

Proof. Assume (1). Suppose K_0 and K_1 are disjoint $\text{PC}(L)$-classes and $\tau = \tau(K_0) = \tau(K_1)$. Let $\varphi_0 \in F$ with $\mathfrak{M} \in K_0$ if and only if $\mathfrak{N} \models \varphi_0$ for some $\mathfrak{N}$ such that $\mathfrak{N} \restriction \tau = \mathfrak{M}$ and $\varphi_1 \in F$ with $\mathfrak{M} \in K_1$ if and only if $\mathfrak{N} \models \varphi_1$ for some $\mathfrak{N}$ such that $\mathfrak{N} \restriction \tau = \mathfrak{M}$. Using the renaming property of L we can change the non-logical symbols of φ_1, other than those in τ, to something completely new, and thereby make sure $\tau = \tau(\varphi_0) \cap \tau(\varphi_1)$. Using closure of L under negation, we can find $\neg\varphi_1 \in F$. Since K_0 and K_1 are disjoint, $\varphi_0 \models \neg\varphi_1$. By (1), there is $\theta \in F'$ such that $\varphi_0 \models \theta$, $\theta \models \varphi_1$ and $\tau(\theta) \subseteq \tau$. Now the $\text{EC}(L')$-class $K = \{\mathfrak{M} \in S : \mathfrak{M} \models \theta\}$ separates K_0 and K_1 i.e. $K_0 \subseteq K$ and $K \cap K_1 = \emptyset$.

Assume (2). Suppose $\varphi \models \psi$, where $\varphi, \psi \in F$. Let $\tau = \tau(\varphi) \cap \tau(\psi)$. Since L is closed under negation, $\neg\psi \in F$. Let $K_0 = \{\mathfrak{M} \restriction \tau : \mathfrak{M} \in S, \mathfrak{M} \models \varphi\}$ and $K_1 = \{\mathfrak{M} \restriction \tau : \mathfrak{M} \in S, \mathfrak{M} \models \neg\psi\}$. Now $K_0 \cap K_1 = \emptyset$. By (2) there is an $\text{EC}(L')$-class K such that $K_0 \subseteq K$, $K \cap K_1 = \emptyset$ and $\tau(K) = \tau$. Let $\theta \in F'$ such that $\tau(\theta) = \tau$ and $K = \{\mathfrak{M} \in S : \tau(\mathfrak{M}) = \tau \text{ and } \mathfrak{M} \models \theta\}$. Now $\varphi \models \theta$, $\theta \models \psi$, and $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$. ◀

The assumption about being closed under negation is essential as the following example from [26] shows: Let L be first order logic, with negation only in front of atomic formulas, added with the generalized quantifier "there are infinitely many x such that ...", i.e. the quantifier Q_0. Every $\text{PC}(L)$-class is $\text{PC}(L_{\omega\omega})$, as can be proved by induction on $\text{PC}(L)$-definitions, and therefore separation holds for L, even $\text{Sep}(L, L_{\omega\omega})$. On the other hand, the proof of Theorem 19 shows that $\text{Craig}(L)$ fails.

A famous consequence of the Craig Interpolation Theorem is the Beth Definability Theorem ([7]), which can be formulated as follows in the abstract logic setting:

▶ **Definition 13.** *Suppose L and L' are classical, atomic and closed under conjunction, and negation.*
1. *We say that $\text{Beth}(L, L')$ holds if every $\varphi \in F$ and every predicate symbol $P \in \tau(\varphi)$ of arity n satisfy: If φ' denotes φ with P renamed as $P' \notin \tau(\varphi)$ and $\varphi \wedge \varphi' \models P(c_1, \ldots, c_n) \leftrightarrow P'(c_1, \ldots, c_n)$, where $c_1, \ldots, c_n \notin \tau(\varphi)$, then there is $\theta \in F'$ such that $P \notin \tau(\theta)$ and $\varphi \models P(c_1, \ldots, c_n) \leftrightarrow \theta$. A logic L has the Beth property, $\text{Beth}(L)$, if $\text{Beth}(L, L)$ holds.*
2. *We say that $\text{WBeth}(L, L')$ holds if the above condition for $\text{Beth}(L, L')$ holds with the additional assumption that every $\mathfrak{M} \in S$ with $\tau(\mathfrak{M}) = \tau(\varphi) \setminus \{P\}$ can be expanded to a model $\mathfrak{N} \in S$ such that $\mathfrak{N} \models \varphi$. A logic L has the weak Beth property ([23]), $\text{WBeth}(L)$, if $\text{WBeth}(L, L)$ holds.*

Clearly, $\text{Beth}(L, L')$ implies $\text{WBeth}(L, L')$. Note also, that $\text{WBeth}(L, L')$ implies $L \leq L'$.

It is interesting to note that the fact that first order logic has the Beth property, was proved by Beth well before the Craig Interpolation Theorem was published. The Beth property is

sometimes formulated with a theory in place of our single sentence φ. The two formulations are equivalent for $L_{\omega\omega}$, thanks to the Compactness Property.

The following establishes an easy connection between interpolation and the Beth property:

▶ **Proposition 14.** *Suppose L and L' are classical abstract logics, closed under conjunction and negation, and $L \leq L'$. Then $\mathrm{Craig}(L, L')$ implies $\mathrm{Beth}(L, L')$.*

Proof. Suppose $\varphi \in F$ such that if φ' denotes φ with P renamed as P', which is not in $\tau(\varphi)$, then $\varphi \wedge \varphi' \models P(c_1, \ldots, c_n) \leftrightarrow P'(c_1, \ldots, c_n)$. Thus $\varphi \wedge P(c_1, \ldots, c_n) \models \varphi' \rightarrow P'(c_1, \ldots, c_n)$. By $\mathrm{Craig}(L, L')$ there is $\theta \in F'$ such that $\varphi \wedge P(c_1, \ldots, c_n) \models \theta$, $\theta \models \varphi' \rightarrow P'(c_1, \ldots, c_n)$, and $\tau(\theta) = (\tau(\varphi) \setminus \{P\}) \cup \{c_1, \ldots, c_n\}$. Then $\varphi \models P(c_1, \ldots, c_n) \leftrightarrow \theta$. ◀

Even compactness does not help to prove the converse: The logic $L(Q^{\mathrm{cof}}_{\leq 2^\omega})$ (see section 3) has an extension (the so-called "Beth-closure" of $L(Q^{\mathrm{cof}}_{\leq 2^\omega})$) which has the Beth property and is compact but it does not satisfy interpolation, not even Souslin-Kleene interpolation [51]. However, there is a stronger form of the Beth property, *projective* $\mathrm{Beth}(L)$, which is actually equivalent to $\mathrm{Craig}(L)$ for regular logics ([2, p. 76]). Interpolation cannot be weakened to Souslin-Kleene interpolation in Proposition 14, as the Δ-closure of $L(Q^{\mathrm{cof}}_{\leq 2^\omega})$ does not have the Beth property [40, 52].

▶ **Definition 15.** **1.** *A classical abstract logic $L = (S, F, \models, \tau)$ is* fully classical *if S consists of* all *structures of first order logic.*
2. *The logic L satisfies* relativization *if for every $\varphi \in F$ and every formula $\psi(x) \in F$ there is a sentence $\theta \in F$ such that for all structures $\mathfrak{M}$ we have $\mathfrak{M} \models \theta$ if and only if the relativization[6] $\mathfrak{M}^{(A)}$ of $\mathfrak{M}$ to A satisfies φ, where A is the set of $a \in M$ such that $\mathfrak{M} \models \psi(a)$, and it is assumed that A is closed under the functions of $\mathfrak{M}$ and also contains the constants of $\mathfrak{M}$.*
3. *The classical logic L satisfies* isomorphism closure, *if $\mathfrak{M} \models \varphi \iff \mathfrak{N} \models \varphi$ whenever $\mathfrak{M}, \mathfrak{N} \in S$, $\mathfrak{M} \cong \mathfrak{N}$, and $\varphi \in F$.*

Examples of fully classical abstract logics satisfying relativization and isomorphism closure are first order logic, $L_{\omega_1\omega}$, second order logic as well as many extensions of first order logic by generalized quantifiers.

▶ **Theorem 16** ([39]). *Assume that the abstract logic L is fully classical, satisfies both relativization and isomorphism closure, and has the property that the vocabulary of each sentence is finite.[7] If L has the (many-sorted) Robinson property then it satisfies the Compactness Theorem.*

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[6] The relativization $\mathfrak{M}^{(A)}$ of $\mathfrak{M}$ to A is the structure which has A as the domain and interpretations of the non-logical symbols as follows: The n-ary relation symbols R in the vocabulary of $\mathfrak{M}$ are interpreted as $R^{\mathfrak{M}} \cap A^n$. The n-ary function symbols f are interpreted as restrictions of $f^{\mathfrak{M}}$ to A^n. Finally, the constant symbols are interpreted in the same way as in $\mathfrak{M}$.

[7] This is called the "Finite Occurrence property". It essentially means that each sentence is a finite string a symbols. It follow easily from compactness.

Proof. We present a rough sketch only and refer to [39] for a detailed proof. We indicate the main idea of proving countable compactness which should give a good idea of the proof in the general case. Suppose $T = \{\varphi_n : n < \omega\}$ is a counter-example to countable compactness. For each n there is $\mathfrak{M}_n \models \{\varphi_m : m < n\}$. W.l.o.g. the domains of the models $\mathfrak{M}_n$ are disjoint and the structures $\mathfrak{M}_n$ are relational. Let $\mathfrak{A}$ be the disjoint union of the models $\mathfrak{M}_n$, $n < \omega$. We expand $\mathfrak{A}$ with a new sort s consisting of a copy of the natural numbers with their natural order $<$ as well as a function f which maps each element of M_n to the n'th natural number in the order $<$. Let $\mathfrak{A}'$ be the expansion. Now in any model $\mathfrak{B}$ that is L-equivalent to $\mathfrak{A}'$ the order-type of $<^{\mathfrak{B}}$ must be ω, for if there were a non-standard number b in the sort s part of B, its pre-image $(f^{\mathfrak{B}})^{-1}(b)$ would determine a model in which each φ_n is true, contrary to the inconsistency of T. Let τ_0 consist of the sort $\{s\}$ and unary predicates P_n, $n < \omega$, of sort s. Let $\mathfrak{A}^*$ be the expansion of $\mathfrak{A}'$ by letting $P_n^{\mathfrak{A}^*}$ consist of the first n elements of $<$ for all $n < \omega$. Let $\mathfrak{A}''$ be a new $\tau_0 \cup \{c\}$-structure, where c is a new constant of sort s, which has $\omega + 1$ as its domain and $P_n^{\mathfrak{A}''} = \{0, \ldots, n-1\}$ as well as $c^{\mathfrak{A}''} = \omega$. By isomorphism closure and the finite occurrence assumption, $\mathfrak{A}^* \restriction \tau_0 \equiv_L \mathfrak{A}'' \restriction \tau_0$. By the Robinson property, there is $\mathfrak{B}$ such that $\mathfrak{B} \restriction \tau(\mathfrak{A}^*) \equiv_L \mathfrak{A}^*$ and $\mathfrak{B} \restriction \tau(\mathfrak{A}'') \equiv_L \mathfrak{A}''$. This contradicts the inconsistency of T, as $(f^{\mathfrak{B}})^{-1}(c^{\mathfrak{B}})$ gives rise to a model of each φ_n. ◀

The assumption of the finite occurrence property in Theorem 16 can be considerably weakened, see [39].

To present an elementary proof of the Robinson property of first order logic, we recall the following game, due to A. Ehrenfeucht [18]: Let L be a finite relational vocabulary and $\mathfrak{A}, \mathfrak{B}$ L-structures such that $A \cap B = \emptyset$. We use $\mathrm{EF}_n(\mathfrak{A}, \mathfrak{B})$ to denote the n-move Ehrenfeucht-Fraïssè game on $\mathfrak{A}$ and $\mathfrak{B}$. During each round of the game player I first picks an element from one of the models, and then player II picks an element from the other model. In this way a relation $p = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq A \times B$ is built. If p is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, player II is the winner of this play. Player II has a winning strategy in this game if and only if the models $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same first order sentences of quantifier rank at most n. If player II has a winning strategy τ in $\mathrm{EF}_n(\mathfrak{A}, \mathfrak{B})$, the set I_i of sets of positions $\{(a_1, b_1), \ldots, (a_j, b_j)\} \subseteq A \times B$ which can be continued to a position $\{(a_1, b_1), \ldots, (a_{n-i}, b_{n-i})\} \subseteq A \times B$ in which player II has used τ, form an increasing chain $I_n \subseteq I_{n-1} \subseteq \ldots \subseteq I_0$ known as a *back-and-forth sequence*, introduced by R. Fraïssè [22]. The name derives from the fact that if $p \in I_{i+1}$, $|p| = k$, and $a \in M$ (or $b \in M'$), then there is $b \in M'$ (respectively, $a \in M$) such that $p \cup \{(a, b)\} \in I_i$. In this case we say that I_{i+1} satisfies the k-back-and-forth condition w.r.t. I_i for k-sequences between the models $\mathfrak{A}$ and $\mathfrak{B}$. Conversely, if a back-and-forth sequence $I_n \subseteq I_{n-1} \subseteq \ldots \subseteq I_0$ exists, then player II can use it to win the game $\mathrm{EF}_n(\mathfrak{A}, \mathfrak{B})$. So back-and-forth sequences and winning strategies of II go hand in hand. This explains why the game is generally called the Ehrenfeucht-Fraïssè game. The infinite version, where players play ω moves, is denoted $\mathrm{EF}(\mathfrak{A}, \mathfrak{B})$.

The following proof of the Robinson property for $L_{\omega\omega}$, using the back-and-forth method just described, is due to Per Lindström (see [36]). This is the argument that became the proof of the so-called Lindström's Theorem ([35]).

▶ **Theorem 17** ([35]). *Suppose L is a regular abstract logic in the sense of [2, Chapter II] such that L is compact, has the Löwenheim property, and has the property that the vocabulary of each sentence is finite.*[8] *Then L has the Robinson property and, in fact, $L \equiv L_{\omega\omega}$.*

Proof. In the Robinson property, suppose Σ_1 and Σ_2 are consistent extensions of a complete theory Σ_0. Let τ_1 be the vocabulary of Σ_1, τ_2 that of Σ_2, and τ that of Σ_0, where $\tau = \tau_1 \cap \tau_2$. By assumption, there is a model $\mathfrak{M}_1$ of Σ_1 and a model $\mathfrak{M}_2$ of Σ_2 and then $\mathfrak{M}_1 \restriction \tau \equiv \mathfrak{M}_2 \restriction \tau$. Thus there is, for any $n \in \mathbb{N}$, a back-and-forth sequence $(I_i : i \leq n)$ for $\mathfrak{M}_1 \restriction \tau$ and $\mathfrak{M}_2 \restriction \tau$. Let τ_2' be a copy of τ_2 such that $\tau_2 \cap \tau_2' = \emptyset$. Let τ' be the vocabulary resulting from τ in this translation. Let $\mathfrak{M}_2'$ be the translation of $\mathfrak{M}_2$ to the vocabulary τ_2'. Let R be a new unary and $<$ a new binary predicate symbol. Let Γ be the set of first order sentences which state:

1. The complete first order τ_1-theory of $\mathfrak{M}_1$.
2. The complete first order τ_2'-theory of $\mathfrak{M}_2$.
3. $(R, <)$ is a non-empty linear order in which every element with a predecessor has an immediate predecessor.
4. First order sentences which state, by means of new $2n + 1$-ary predicates I^n, that there is, for each $n < \omega$, an n-back-and-forth sequence $I^n(i, \cdot, \cdot)$, parametrized by $i \in R$, between the τ-reduct and the τ'-reduct of the universe, meaning that if $i - 1$ denotes the predecessor (if it exists) of i in R, then $I^n(i, \cdot, \cdot)$ satisfies the n-back-and-forth condition w.r.t. $I^{n+1}(i - 1, \cdot, \cdot)$ for n-sequences between the τ-reduct and the τ'-reduct of the universe.

For all $n \in \mathbb{N}$ there is a model of Γ with $(R, <)$ of length n. By the Compactness Theorem, there is a model $\mathfrak{N}$ of Γ with $(R, <)$ non-well-founded. By the Löwenheim property we may assume $\mathfrak{N}$ is countable. Let $\mathfrak{N}_1 = \mathfrak{N} \restriction \tau_1$ and $\mathfrak{N}_2 = \mathfrak{N} \restriction \tau_2'$. Since $(R, <)^{\mathfrak{N}}$ is non-well-founded, player II has a winning strategy for the infinite game $\mathrm{EF}(\mathfrak{N}_1, \mathfrak{N}_2)$. Since $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are countable, $\mathfrak{N}_1 \cong \mathfrak{N}_2$. This implies that $\mathfrak{N}_1$ can be expanded to a model of $\Sigma_1 \cup \Sigma_2$. This ends the proof of Robinson property.

Lindström observed that the above argument works for any logic which is countably compact and has the Löwenheim property. But are there such extensions of $L_{\omega\omega}$? To see why there are none, let us write $\mathfrak{M} \sim_n \mathfrak{M}'$ if player II has a winning strategy in $\mathrm{EF}_n(\mathfrak{A}, \mathfrak{B})$. This equivalence relation divides the class of all models with a finite vocabulary L into a finite number of equivalence classes C, each definable by a first order sentence of quantifier rank at most n, namely the conjunction of all first order sentence of quantifier rank at most n that

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[8] This actually follows from compactness with an easy argument.

are true in one (hence all) models in C. Suppose now $\varphi \in F$ is not first order definable, i.e. there is no first order ψ such that $\varphi \equiv \psi$ and $\tau(\varphi) = \tau(\psi)$. By our assumption, $\tau(\varphi)$ is finite. There is no n such that the class of models of φ is closed under $\sim_n$. In other words, for all n there are models $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that $\mathfrak{M}_n \sim_n \mathfrak{N}_n$, $\mathfrak{M}_n \models \varphi$ and $\mathfrak{N}_n \models \neg\varphi$. By countable compactness there are elementarily equivalent $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \neg\varphi$. Now we can continue as above and obtain two isomorphic models, one of φ and the other of $\neg\varphi$, a contradiction. ◀

An amusing corollary of the above Lindström Theorem is that $L_{\omega\omega}$ satisfies the Souslin-Kleene Interpolation theorem: Consider $\Delta(L_{\omega\omega})$. It is a regular logic, compact and has the Löwenheim property. Hence $\Delta(L_{\omega\omega}) \equiv L_{\omega\omega}$. There is no similarly easy way (as far as is known) to deduce the Beth property or the Craig Interpolation property directly from Lindström's Theorem.

The proof of Theorem 17 actually gives the following slightly more general result:

▶ **Theorem 18** ([20], see also [26]). *Suppose L is a regular abstract logic, except that L is not assumed to be closed under negation. If L is compact, satisfies the Löwenheim property, and has the property that the vocabulary of each sentence is finite, then any two disjoint $EC(L)$-classes can be separated by an $EC(L_{\omega\omega})$-class.*

For Lindström Theorems for fragments of first order logic we refer to [6].

3 Generalized Quantifiers

Sometimes one may want a logic that can express some particular property, be it cardinality, cofinality, connectedness, well-foundedness, or some other structural property. One way to accomplish this is by means of the concept of a generalized quantifier, introduced in [46] and extended in [34]. Generalized quantifiers allow one to add a particular feature to a logic. Unsurprisingly, there is no guarantee that such an addition results in a nice logic, e.g. a logic with interpolation. To obtain a nice logic one probably has to add many generalized quantifiers. But then another problem arises. Any logic (closed under substitution) can be represented as the result of adding a number of generalized quantifiers to first order logic. Since it is unlikely that every logic is "nice" in any sense, it makes sense to limit oneself to adding just one or finitely many generalized quantifiers.

The first generalized quantifier to be considered (in [46]) was

$$Q_\alpha x \varphi(x) \iff |\{a : \varphi(a)\}| \geq \aleph_\alpha.$$

Here α is a fixed ordinal chosen in advance. The best-known cases are $\alpha = 0$ or $\alpha = 1$. The extension $L(Q_\alpha)$ of first order logic by this quantifier has the Löwenheim property for $\alpha = 0$ and is countably compact for $\alpha = 1$. The following simple result, due to Keisler, is from [37]:

▶ **Theorem 19.** $L(Q_\alpha)$ *does not have the interpolation property.*

Proof. Let E be a binary relation symbol and S, S' unary relation symbols. Let φ the conjunction of

(1) $\forall x(xEx) \wedge \forall x \forall y(xEy \to yEx) \wedge \forall x \forall y \forall z((xEy \wedge yEz) \to xEz))$.

(2) $\forall x \exists y(xEy \wedge S(y))$.

(3) $\forall x \forall y((S(x) \wedge S(y) \wedge xEy) \to x = y)$.

(4) $Q_\alpha x S(x)$.

Let ψ the sentence

(5) $\forall x \exists y(S'(y) \wedge xEy) \to Q_\alpha x S'(x)$.

Clearly, $\varphi \models \psi$. Suppose $\varphi \models \theta$ and $\theta \models \psi$, where $\theta \in L(Q_1)$ has vocabulary $\{E\}$ only. Then θ says that E is an equivalence relation with at least $\aleph_\alpha$ equivalence classes. But we now use an Ehrenfeucht-Fraïssè game to show that such a θ cannot exist. In the Ehrenfeucht-Fraïssè game of $L(Q_\alpha)$ on two models $\mathfrak{M}$ and $\mathfrak{N}$ players build a partial isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. Suppose a partial isomorphism p has been built. In addition to the usual moves of the Ehrenfeucht-Fraïssè game of first order logic, player I has the option of choosing a subset X of cardinality $\geq \aleph_\alpha$ of one of the models, say $\mathfrak{M}$, after which player II chooses a subset Y of size $\geq \aleph_\alpha$ of the other model, in this case $\mathfrak{N}$. After this player I chooses an element $y \in Y$ and player II responds by choosing an element $x \in X$. The game continues now with the partial mapping $p \cup \{(x, y)\}$ while X and Y are abandoned. It is not hard to show that if II has a winning strategy in such a game of any finite length, then the models are $L(Q_\alpha)$-equivalent (see e.g. [56, Chapter 10]). For the current case we can use a model $\mathfrak{M}$ which is an equivalence relation with $\aleph_\alpha$ classes, each of them of size $\aleph_\alpha$. For $\mathfrak{N}$ we can use an equivalence relation with $\aleph_0$ classes, each of size $\aleph_\alpha$. The winning strategy of player II is the following. Suppose player I has played a set $X \subseteq M$, and player II now has to choose the set $Y \subseteq N$. Let A be the union of all equivalence classes of elements of dom(p), and B the union of all equivalence classes of elements of ran(p). If $X \setminus A \neq \emptyset$, we let II choose $N \setminus B$. This is clearly a move that keeps II in the game. We can therefore assume w.l.o.g. that $X \subseteq A$. Since $X \setminus \text{dom(p)}$ is infinite, there is $a \in \text{dom(p)}$ such that $X \setminus \text{dom(p)}$ meets the equivalence class of a. Now we let II choose Y to be what is in the equivalence class of $p(a)$ outside ran(p). This is clearly a winning strategy for any finite number of moves. This contradicts the fact that θ separates $\mathfrak{M}$ and $\mathfrak{N}$.   ◄

The proof also shows that even Souslin-Kleene Interpolation fails for $L(Q_\alpha)$. In fact, the proof shows[9] that $\Delta(L(Q_\alpha)) \not\leq L_{\infty\omega}(Q_\alpha)$. The Beth theorem fails for $L(Q_1)$ ([23]), but surprisingly, it is consistent, relative to the consistency of ZF, that the weak Beth property holds for $L(Q_1)$ ([45]).

▶ **Theorem 20** ([47]). *$L(Q_0)$ does not have the weak Beth property.*

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[9] This is because the strategy of player II works not only for any finite game but even for the game of length ω.

Proof. The crucial property of $L(Q_0)$ here is that it can express "every natural number has only finitely many predecessors". Let $\varphi \in L(Q_0)$ say of its models (A, E, R) that (A, E) is a model of a suitable finite part T of ZFC and either the natural numbers of (A, E) have non-standard elements[10] and $R = \emptyset$ or the natural numbers of (A, E) are standard[11] and R is the set of pairs (η, f), where (A, E) satisfies

1. $\eta \in L(Q_0)$, and

2. f is a function such that the inductive clauses for satisfaction of $L(Q_0)$-formulae of the vocabulary $\{E\}$ hold.

An example of the inductive clauses here is:

- $(Q_0 x \eta, f) \in R$ if and only if for infinitely many y there exists $g \in A$ such that $g(z) = f(y)$ for variables $z \neq x$, $g(x) = y$ and $(\eta, g) \in R$.

If (A, E, R) and (A, E, R') are models of φ, and (w.l.o.g.) the integers of (A, E) are all standard, then one can use induction on formulas of $L(Q_0)$ of the vocabulary $\{E\}$ to prove that $R = R'$. Moreover, for all (A, E) there is always an R such that $(A, E, R) \models \varphi$. Suppose that there were a formula $\eta(x, y) \in L(Q_0)$ which defines R explicitly in models of φ. Let $\kappa > \omega$ be a cardinal such that $\mathfrak{M} = (H_\kappa, \in)$, where H_κ is the set of sets whose transitive closure has cardinality $< \kappa$, is a model of T. If now R is chosen such that $(H_\kappa, \in, R) \models \varphi$, then

$$R = \{(\varphi, f) : \varphi \in L(Q_0) \text{ and } f \text{ satisfies } \varphi \text{ in } \mathfrak{M}\}.$$

Combining this with the choice of $\eta(x, y)$ yields

$$\mathfrak{M} \models \eta(\varphi, f) \text{ if and only if } f \text{ satisfies } \varphi \text{ in } \mathfrak{M}.$$

The standard diagonal argument ends the proof. To this end, let $\xi (= \xi(x))$ be the formula $\neg \eta(x, f)$, where f is a term denoting the function which maps the variable x to x ($f = \{(x, x)\}$). We now have $\xi \in H_\kappa$ and

$$\mathfrak{M} \models \xi(\xi) \leftrightarrow \eta(\xi, \{(x, x)\}) \leftrightarrow \neg \xi(\xi).$$

This contradiction shows that φ constitutes a counter-example to the weak Beth property for the logic $L(Q_0)$.   ◀

The point of the above proof is that with $L(Q_0)$ we can capture the standard natural numbers and thereby use induction on objects, which are formulas "merely" in the sense of the model (A, E). Otherwise it does not matter what the new quantifier Q_0 says. The same proof gives a result about any generalized quantifiers $Q^1, \ldots, Q^n$ in the sense of [34]: If $L(Q^1, \ldots, Q^n) \not\equiv L_{\omega\omega}$ has Löwenheim property, then the weak Beth property fails ([35]).

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[10] I.e. elements which have infinitely many predecessors. This is where we use Q_0.

[11] I.e. they all have only finitely many predecessors. This is again where we use Q_0.

If we look how much we have to add to $L(Q_0)$ to obtain weak Beth (or even interpolation), the optimal answer is $\text{WBeth}(L(Q_0), L_{\text{HYP}})$, where HYP is the smallest admissible set containing ω (see [1] for details) and $L_{\text{HYP}} = L_{\omega_1 \omega} \cap \text{HYP}$. Thus we have to resort to infinitary propositional operations to obtain Beth definability for $L(Q_0)$, and then we get full interpolation as a bonus.

If we want interpolation for a logic of the form $L(Q^1, \ldots, Q^n)$, we need to have countable (or at least "recursive") compactness. However, no proper extension of first order logic is known which has both interpolation and countable compactness.

Problem: Are FO quantifiers the only generalized quantifiers which give rise to a regular logic with the Craig Interpolation property? The same for the Beth property and Souslin-Kleene. Does weak Beth hold for $L(Q_1)$ provably in ZFC?

Problem: Is there any proper extension of first order logic that is (countably) compact and has the interpolation property?

One of the most notorious generalized quantifiers is the cofinality quantifier Q_κ^{cof} ([50]). For a regular κ it is defined as follows:

$$\mathfrak{M} \models Q_\kappa^{\text{cof}} xy\varphi(x, y, \vec{a}) \quad \Longleftrightarrow \quad \{(c, d) : \mathfrak{M} \models \varphi(c, d, \vec{a})\}$$
$$\text{is a linear order of cofinality } \kappa.$$

What is remarkable about this quantifier is that it is compact in a vocabulary of any cardinality, which is why it is called "fully compact". It has also a nice complete axiomatization. However, the logic $L(Q_\omega^{\text{cof}})$ does not have interpolation or Beth properties ([41]). Still, interestingly, there is a countably compact logic $L(aa)$ with $\text{Craig}(L(Q_\omega^{\text{cof}}), L(aa))$, see Theorem 27.

4   Infinitary Logics

Infinitary logics, such as $L_{\omega_1 \omega}$, are based on a completely different idea than generalized quantifiers. Here we add new logical operations[12] which resemble the familiar logical operations of $L_{\omega\omega}$ but are "infinitary". While in the case of generalized quantifiers we did not mind if the meaning of formulas became "infinitary", as in "there are infinitely many" and "there are uncountably many", here we do not mind if the formulas themselves become syntactically "infinitary", as in $\varphi_0 \vee \varphi_1 \vee \ldots$. The advantage of this change of perspective is that we obtain lots of examples of extensions of first order logic with interpolation. Since we lose compactness (apart from so-called Barwise compactness), we cannot obtain the Robinson property, the model theoretic version of interpolation. The first proofs of interpolation for infinitary logics were indeed proof-theoretic.

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[12] In $L_{\kappa\lambda}$ conjunctions and disjunctions of sets of formulas of size $< \kappa$ are allowed as well as quatification of sequences of variables of length $< \lambda$. Then $L_{\infty\lambda}$ is $\bigcup_\kappa L_{\kappa\lambda}$ and $L_{\infty\infty}$ is $\bigcup_\lambda L_{\infty\lambda}$.

A paradigmatic example of an $L_{\omega_1\omega}$-sentence is $\forall x \bigvee_n (x = s^n(0))$ which says of a unary function s and a constant 0 that every element is either 0 or obtained from 0 by iterating the function s, taking $s^0(0)$ to be 0.

We embrace now fully the many-sorted approach to logic as it yields a particularly powerful form of interpolation. Thus we have variables x^s of different sorts s. Let sort(τ) denote the set of sorts of the vocabulary τ.

Let us recall the auxiliary concept of a Hintikka set: Suppose τ is countable, C^s is a countable set of new constant symbols for $s \in$ sort(τ), and $\tau' = \tau \cup C^*$, where $C^* = \bigcup_s C^s$. A *Hintikka set* is any set H of τ'-sentence of $L_{\omega_1\omega}$, which satisfies:

1. $t = t \in H$ for every constant τ'-term t.
2. If $\varphi(t) \in H$, $\varphi(t)$ atomic, and $t = t' \in H$, then $\varphi(t') \in H$.
3. If $\neg\varphi \in H$, then $\varphi\neg \in H$.[13]
4. If $\bigvee_n \varphi_n \in H$, then $\varphi_n \in H$ for some n.
5. If $\bigwedge_n \varphi_n \in H$, then $\varphi_n \in H$ for all n.
6. If $\exists x^s \varphi(x^s) \in H$, then $\varphi(c) \in H$ for some $c \in C^s$.
7. If $\forall x^s \varphi(x^s) \in H$, then $\varphi(c) \in H$ for all $c \in C^s$.
8. For every constant τ'-term t of sort s there is $c \in C^s$ such that $t = c \in H$.
9. There is no atomic φ such that $\varphi \in H$ and $\neg\varphi \in H$.

The Hintikka set H is a Hintikka set *for* a sentence φ of $L_{\omega_1\omega}$ (or $L_{\omega\omega}$) if $\varphi \in H$. The basic property of Hintikka sets is that if $\varphi \in L_{\omega_1\omega}$ does have a model $\mathfrak{M}$, then there is a Hintikka set for φ, built directly from formulas true in $\mathfrak{M}$; and conversely, if there is a Hintikka set H for a given $\varphi \in L_{\omega_1\omega}$, then φ has a model, as we shall now see. A model $\mathfrak{M}$ is built from H as follows: Define on $C^* = \bigcup_s C^s$ an equivalence relation $c \sim c'$ by the condition $c = c' \in H$. Let the domain of sort s in $\mathfrak{M}$ be $M_s = \{[c] : c \in C^s\}$. The interpretations in $\mathfrak{M}$ are defined by $c^{\mathfrak{M}} = [c]$, $f^{\mathfrak{M}}([c_{i_1}], \ldots, [c_{i_n}]) = [c]$ for $c \in C^*$ such that $f(c_{i_1}, \ldots, c_{i_n}) = c \in H$. For any constant term t of sort s there is a $c \in C^s$ such that $t = c \in H$. It is easy to see that $t^{\mathfrak{M}} = [c]$. If R is an n-ary predicate symbol, we let $(t_1, \ldots, t_n) \in R^{\mathfrak{M}}$ if and only if $R(t_1, \ldots, t_n) \in H$. By induction on $\varphi(x_1, \ldots, x_n)$ one can now easily prove that if $d_1 \ldots, d_n \in C^*$ then,

$$\varphi(d_1, \ldots, d_n) \in H \quad \Rightarrow \quad \mathfrak{M} \models \varphi(d_1, \ldots, d_n)$$
$$\neg\varphi(d_1, \ldots, d_n) \in H \quad \Rightarrow \quad \mathfrak{M} \not\models \varphi(d_1, \ldots, d_n).$$

In particular, $\mathfrak{M} \models \varphi$ for the φ we started with, since $\varphi \in H$.

How do we find useful Hintikka sets? The basic tool is the auxiliary concept of a consistency property. Roughly speaking, a consistency property is a set Δ of (usually) finite sets S which are consistent and Δ has information about how to extend S to a Hintikka set.

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[13] $\varphi\neg$ is $\neg\varphi$, if φ is atomic, $(\neg\varphi)\neg$ is φ, $(\bigwedge_n \varphi_n)\neg$ is $\bigvee_n \neg\varphi_n$, $(\bigvee_n \varphi_n)\neg$ is $\bigwedge_n \neg\varphi_n$, $(\forall x^s \varphi)\neg$ is $\exists x^s \neg\varphi$, and $(\exists x^s \varphi)\neg$ is $\forall x^s \neg\varphi$

A *consistency property* is any set Δ of countable sets S of τ-formulas of $L_{\omega_1\omega}$, which satisfies the conditions:

1. If $S \in \Delta$, then $S \cup \{t = t\} \in \Delta$ for every constant τ'-term t.
2. If $\varphi(t) \in S \in \Delta$, $\varphi(t)$ atomic, and $t = t' \in S$, then $S \cup \{\varphi(t')\} \in \Delta$.
3. If $\neg\varphi \in S \in \Delta$, then $S \cup \{\varphi\neg\} \in \Delta$.
4. If $\bigvee_n \varphi_n \in S \in \Delta$, then $S \cup \{\varphi_n\} \in \Delta$ for some n.
5. If $\bigwedge_n \varphi_n \in S \in \Delta$, then $S \cup \{\varphi_n\} \in \Delta$ for all n.
6. If $\exists x^s \varphi(x^s) \in S \in \Delta$, then $S \cup \{\varphi(c)\} \in \Delta$ for some $c \in C^s$.
7. If $\forall x^s \varphi(x^s) \in S \in \Delta$, then $S \cup \{\varphi(c)\} \in \Delta$ for all $c \in C^s$.
8. For every constant L'-term t of sort s there is $c \in C^s$ such that $S \cup \{t = c\} \in \Delta$.
9. There is no atomic formula φ such that $\varphi \in S$ and $\neg\varphi \in S$.

The consistency property Δ is a consistency property *for* a set T of infinitary τ-sentences if for all $S \in \Delta$ and all $\varphi \in T$ we have $S \cup \{\varphi\} \in \Delta$.

A *τ-fragment* of $L_{\omega_1\omega}$ is any set $\mathcal{F}$ of formulas of $L_{\omega_1\omega}$ in the vocabulary τ such that $\mathcal{F}$ is closed under substitutions of terms, $\mathcal{F}$ contains the atomic τ-formulas, $\neg\varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$, $\wedge\Phi \in \mathcal{F}$ if $\Phi \subseteq \mathcal{F}$ is finite, $\vee\Phi \in \mathcal{F}$ if $\Phi \subseteq \mathcal{F}$, is finite, $\wedge\Phi \in \mathcal{F}$ if and only if $\vee\Phi \in \mathcal{F}$, and then $\Phi \subseteq \mathcal{F}$, $\forall x^s \varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$, $\exists x^s \varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$. Note that a fragment is necessarily closed under subformulas. It is easy to see that if $\varphi \in L_{\omega_1\omega}$ with a countable vocabulary τ, then there is a countable fragment $\mathcal{F} \subseteq L_{\omega_1\omega}$ such that $\varphi \in \mathcal{F}$.

▶ **Lemma 21.** *Let T be a countable set of τ-sentences of $L_{\omega_1\omega}$ or $L_{\omega\omega}$ and $\mathcal{F} \subseteq L_{\omega_1\omega}$ a countable fragment such that $T \subseteq \mathcal{F}$. Suppose Δ is a consistency property for T. Then for any $S \in \Delta$ there is a Hintikka set H for T such that $S \subseteq H$.*

Proof. We can define H as the union of an increasing sequence S_n, $n < \omega$, where $S_0 = S$. The sequence is constructed straightforwardly in such a way, maintaining judicious bookkeeping, that in the end the set H is a Hintikka set.   ◀

Let $\mathrm{Un}(\varphi)$ be all sorts s such that a variable of sort s occurs universally quantified in φ. Similarly $\mathrm{Un}(S)$ for a set S of formulas. Let $\mathrm{Ex}(\varphi)$ be all sorts s such that a variable of sort s occurs existentially quantified in φ. Similarly $\mathrm{Ex}(S)$ for a set S of formulas. For example, suppose φ is $\forall x^1 \exists x^0 (x^0 = x^1)$. Then $\mathrm{Un}(\varphi) = \{1\}$ and $\mathrm{Ex}(\varphi) = \{0\}$. Suppose φ is $\forall x^0 \forall y^3 (R(x^0, y^3) \leftrightarrow R'(x^0, y^3))$. Then $\mathrm{Un}(\varphi) = \{0, 3\}$ and $\mathrm{Ex}(\varphi) = \emptyset$.

▶ **Theorem 22** ([19]). *Suppose $\varphi \models \psi$, where φ and ψ are sentences of $L_{\omega_1\omega}$ in a relational vocabulary. Then there is a sentence θ of $L_{\omega_1\omega}$ such that*
1. *$\varphi \models \theta$ and $\theta \models \psi$*
2. *$\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$*
3. *$\mathrm{Un}(\theta) \subseteq \mathrm{Un}(\varphi)$ and $\mathrm{Ex}(\theta) \subseteq \mathrm{Ex}(\psi)$.*

Proof. Let us assume that the claim of the theorem is false and derive a contradiction. Since $\varphi \models \psi$, the set $\{\varphi, \neg\psi\}$ has no models. We construct a consistency property for $\{\varphi, \neg\psi\}$. It then follows that $\{\varphi, \neg\psi\}$ has a model, which is a contradiction. Let $\tau_1 = \tau(\varphi)$, $\tau_2 = \tau(\psi)$, and $\tau = \tau_1 \cap \tau_2$. Suppose $C^s = \{c_n^s : n \in \mathbb{N}\}$ is a set of new constant symbols for each sort s of $\tau_1 \cup \tau_2$. Let $C^* = \bigcup_s C^s$. Given a set S of sentences, let S_1 consists of all $\tau_1 \cup C^*$-sentences in S with only finitely many constant from C^*, and let S_2 consists of all $\tau_2 \cup C^*$-sentences in S with only finitely many constant from C^*. Let us say that θ *separates* S' and S'' if

1. $S' \models \theta$,
2. $S'' \models \neg\theta$,
3. $\mathrm{Un}'(\theta) \subseteq \mathrm{Un}(S')$,
4. $\mathrm{Ex}'(\theta) \subseteq \mathrm{Un}(S'')$,

where $\mathrm{Un}'(\theta)$ consists of sorts $s \in \mathrm{Un}(\theta)$ and sorts of constants $c \in C^*$ occurring in θ, and $\mathrm{Ex}'(\theta)$ consists of sorts $s \in \mathrm{Ex}(\theta)$ and sorts of constants $c \in C^*$ occurring in θ.

Let Δ consist of finite sets S of sentences of $L_{\omega_1\omega}$ such that $S = S_1 \cup S_2$ and:

($\star$) There is no $L \cup C^*$-sentence that separates S_1 and S_2.

Note that $\{\varphi, \neg\psi\} \in \Delta$. We show that Δ is a consistency property. This involves checking the nine conditions that a consistency property has to satisfy. Most are almost trivial. For example, let us look at condition 7. Consider $S \in \Delta$ and $\exists x^s \varphi(x^s) \in S_1$. Let $c_0 \in C^s$ be such that c_0 does not occur in S. Now the sets $S_1 \cup \{\varphi(c_0)\}$ and S_2 satisfy ($\star$). On the other hand, consider $S \in \Delta$ and $\exists x^s \varphi(x^s) \in S_2$. Let again $c_0 \in C^s$ be such that c_0 does not occur in S. Now the sets S_1 and $S_2 \cup \{\varphi(c_0)\}$ satisfy ($\star$). Theorem 22 is proved. ◄

If above $\varphi, \psi \in A$, where A is a countable admissible set, then also $\theta \in A$ [1].

Many sorted interpolation gives numerous preservation results of which we mention just one. It should be noted that we are dealing with infinitary logic, so we do not have a compactness theorem. Also, for example, it is not true that every formula has a prenex normal form ([16]). On the other hand, Theorem 22 gives also many-sorted interpolation for first order logic, so it can be used to prove preservation results for first order logic as well.

Here is an example:

▶ **Theorem 23** ([43]). *A formula φ of $L_{\omega_1\omega}$ is preserved by submodels if and only if it is logically equivalent to a universal formula.*

Proof. Let us assume the single sorted φ is written in sort 0 variables and has just one binary predicate symbol R. Let φ' be the same written in sort 1 variables and with R replaced by R'. Let EXT be the conjunction

$$\forall x^1 \exists x^0 (x^0 = x^1) \wedge \forall x^1 \forall y^1 (R'(x^1, y^1) \leftrightarrow R(x^1, y^1)). \tag{1}$$

Note that $(\{M_0, M_1\}, R, R') \models \forall x^1 \exists x^0 (x^0 = x^1)$ iff $M_1 \subseteq M_0$. The sentence (1) is true in $(\{M_0, M_1\}, R, R')$ iff $(M_1, R') \subseteq (M_0, R)$. By assumption, $\mathrm{EXT} \wedge \neg\varphi' \models \neg\varphi$. Let $\theta \in L_{\omega_1\omega}$ be given by Theorem 22. Thus $\mathrm{EXT} \wedge \neg\varphi' \models \theta$ and $\theta \models \neg\varphi$. The only common sort is 0,

so θ has only sort 0 symbols. To see that θ is existential we note that if a sort 0 variable was universally quantified in θ, then it is universally quantified in $\text{EXT} \wedge \neg\varphi'$, but there is no universally quantified sort 0 variable in $\text{EXT} \wedge \neg\varphi'$. Thus θ is existential. Moreover, $\models \neg\varphi \leftrightarrow \theta$. ◀

The infinitary logic $L_{\omega_1\omega}$ can be extended by new logical operations preserving its "good" properties such as interpolation: Let us call any function $\mathcal{P}(\omega) \to 2$ a *propositional connective*. If P is a propositional connective, then $L_{\omega_1\omega}(P)$ is the extension of $L_{\omega_1\omega}$ by the connective

$$\mathfrak{M} \models_P C(\langle \varphi_i : i < \omega \rangle)(\vec{a}) \iff P(\{i : \mathfrak{M} \models_P \varphi_i(\vec{a})\}) = 1.$$

Harrington [28] proved that there are 2^{2^ω} propositional connectives P such that $L_{\omega_1\omega}(P)$ satisfies the Craig Interpolation Theorem as well as a form of the Barwise Compactness Theorem. Unfortunately Harrington's new propositional connectives are just abstract functions with no intuitive or natural meaning. Still their existence demonstrates that $L_{\omega_1\omega}$ is by no means maximal with respect to the Craig Interpolation Theorem, even if a weak form of compactness is added.

The situation changes radically when we move to $L_{\omega_2\omega}$:

▶ **Theorem 24** ([44]). $\text{Craig}(L_{\omega_2\omega}, L_{\infty\omega})$ *fails.*

Proof. Let $\varphi \in L_{\omega_2\omega}$ say "$<$ is a linear order of order-type ω_1" [49]. Let $\psi \in L_{\omega_1\omega}$ say "$<'$ is a total linear order of order-type ω". Clearly $\varphi \models \neg\psi$. Suppose $\theta \in L_{\infty\omega}$ is such that $\varphi \models \theta$, $\theta \models \neg\psi$ and the vocabulary of θ is $\emptyset$. Then $(\omega_1, <) \models \varphi$, so $(\omega_1) \models \theta$. But $(\omega_1) \equiv_{\infty\omega} (\omega)$, as an easy application of the Ehrenfeucht-Fraïssè game of length ω shows. So $(\omega) \models \theta$, a contradiction. ◀

The method of proof of Theorem 20, i.e. essentially an undefinability of truth argument, can be used to prove:

▶ **Theorem 25** ([27]). **1.** $\text{WBeth}(L_{\omega_2\omega}, L_{\omega_2\omega_2})$ *fails.*
2. $\text{WBeth}(L_{\omega_1\omega_1}, L_{\infty\infty})$ *fails.*

However, with an appropriate modification of the method of the proof of Theorem 22 one can show $\text{Craig}(L_{\kappa\omega}, L_{(2^{<\kappa})^+\kappa})$, where κ is regular ([44]). By modifying the question of interpolation suitably, one can obtain a more balanced result, see [5]. The result $\text{Craig}(L_{\kappa\omega}, L_{(2^{<\kappa})^+\kappa})$ raises the question, whether or not there is a logic L such that $L_{\kappa\omega} \leq L \leq L_{(2^{<\kappa})^+\kappa}$ and L has interpolation? Shelah's new infinitary logic L^1_κ gives one answer to this question ([53]).

To see this, let us redefine $L_{\kappa\omega}$ as $\bigcup_{\lambda<\kappa} L_{\lambda^+\omega}$ and $L_{\kappa\kappa}$ as $\bigcup_{\lambda<\kappa} L_{\lambda^+\lambda^+}$ in the case that κ is a limit cardinal. For regular limits this agrees with the old notation for $L_{\kappa\omega}$ and $L_{\kappa\kappa}$, but for singular cardinals the new notation seems more canonical. Let $\kappa = \beth_\kappa$, where $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\nu = \sup\{\beth_\alpha : \alpha < \nu\}$ for limit ordinals ν. The new infinitary logic L^1_κ introduced in [53] satisfies $L_{\kappa\omega} \leq L^1_\kappa$, $L^1_\kappa \leq L_{\kappa\kappa}$ and $\text{Craig}(L^1_\kappa)$. Moreover, L^1_κ has a

Lindström style model theoretic characterization in terms of a strong form of undefinability of well-order.

The main ingredient of L_κ^1 is a new variant $G_\theta^\beta(A, B)$, where β is an ordinal and θ a cardinal, of the Ehrenfeucht-Fraïssè game for $L_{\kappa\kappa}$. In this variant player II gives only partial answers to moves of player I. She makes promises and fulfills the promises move by move. If the game lasted for ω moves, the partial answers of II would constitute full answers. But the game has a well-founded clock, so II never ends up fully completing her answers. Although L_κ^1 cannot express well-ordering, it can express e.g. the property of a linear order of not having an uncountable descending chain.

Shelah's game $G_\theta^\beta(A, B)$ proceeds as follows:

- At first player I picks $\beta_0 < \beta$ and $\vec{a^0}$ from A with $len(\vec{a^0}) \leq \theta$. The move β_0 is a "clock" move which controls the (finite) length of the game. The move $\vec{a^0}$ is player I's challenge to player II. He wants to know what are the images of the $\leq \theta$ elements of the sequence $\vec{a^0}$.
- Next II picks $f_0 : \vec{a^0} \to \omega$ and $g_0 : A \to B$ a partial isomorphism such that $f_0^{-1}(0) \subseteq$ dom(g_0). Here player II responds to the challenge made by player I. But now comes the catch. Player II divides the set $\vec{a^0}$ into ω pieces with f_0. She is not going to respond yet to the entire challenge $\vec{a^0}$, only to a piece, namely $f_0^{-1}(0)$. To the other pieces she is going to respond later, one by one. With good luck the game ends before she has to respond to $f_0^{-1}(10^{10})$!
- Next I picks $\beta_1 < \beta_0$ and $\vec{b^1}$ from B such that $len(\vec{b^1}) \leq \theta$. Here I challenges II to give the preimages of the $\leq \theta$ elements of $\vec{b^1}$.
- Next II picks $f_1 : \vec{b^1} \to \omega$ and $g_1 : A \to B$ a partial isomorphism, $g_1 \supseteq g_0$ such that $f_0^{-1}(1) \subseteq$ dom(g_1) and $f_1^{-1}(0) \subseteq$ ran(g_1). With f_1 player II splits the challenge $\vec{b^1}$ into ω pieces to which she is going to respond one by one while the game continues. With g_1 she starts responding to the challenge $\vec{b^1}$. In order for her responses to remain consistent, it is necessary that $g_1 \supseteq g_0$. By making sure that $f_1^{-1}(0) \subseteq$ ran(g_1) she responds to the challenge $f_1^{-1}(0)$. Now she has to also continue responding to the challenge $\vec{a^0}$ by making sure $f_0^{-1}(1) \subseteq$ dom(g_1).
- And so on until $\beta_n = 0$.

Player II wins if she can play all her moves, otherwise Player I wins. $A \sim_\theta^\beta B$ if Player II has a winning strategy in the game. $A \equiv_\theta^\beta B$ is defined as the transitive closure of $A \sim_\theta^\beta B$. A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv_\theta^\beta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L_κ^1.

How does this definition make L_κ^1 an abstract logic? One has to work a bit to prove that all the closure properties that we require of a logic are satisfied. In the end, L_κ^1 is an abstract logic in the sense of Definition 1.1.1 of [2]. But what is the syntax like? All we know about the sentences of L_κ^1 is that they are classes of models. Still we can prove:

▶ **Theorem 26** (Shelah [53]). *Suppose $\kappa = \beth_\kappa$. Then L_κ^1 satisfies the Craig Interpolation Theorem.*

The proof is based entirely on properties of the game $G_\theta^\beta(A, B)$ and is not totally unlike the proof of Theorem 17. It is somewhat surprising that many model theoretic results can be proved for L_κ^1 although we are not able (yet) to answer the question:

Problem: What is the syntax of Shelah's logic?

Cartagena logic [33] is a syntactic fragment of Shelah's logic. Its Δ-extension is Shelah's logic L_κ^1. For more on L_κ^1, see [58].

5   Higher Order Logics

Higher order logic are the oldest extensions of first order logic. In *weak second order logic* L_w^2 there are variables for individuals, as in first order logic, but also variables for finite sets of individuals. If X is a set-variable and t is a term denoting an individual, then we can form the atomic formula $X(t)$. Quantifiers range over individual and over set variables. The logic L_w^2 is quite close to the logic $L(Q_0)$ (see section 3), in fact, $\Delta(L_w^2) \equiv \Delta(L(Q_0))$.

It was shown in [47] that the Beth property fails for *weak* second order logic, and for *monadic* second order logic in which second order variables range over arbitrary subsets of the domain but vocabularies are canbe non-monadic. The former claim is proved by means of an undefinability of truth argument, almost verbatim as we did in the proof of Theorem 20, while the latter claim is proved by reduction to the decidable monadic second order theory of the successor function ([8]). Namely, in the standard model of the successor function, which is definable in monadic second order logic, addition and multiplication are implicitly definable by their familiar recursive definitions. If they were explicitly definable, arithmetic would be reducible to the monadic second order theory of the successor function. However, this is impossible because the former is non-arithmetic but the latter is decidable by [8].

In *stationary logic* $L(aa)$, introduced in [50] and [3], we have variables for individuals and also for countable sets of individuals. If s is a set-variable and t is a term denoting an individual, then we can again form the atomic formula $s(t)$. This time we do not have existential and universal quantifiers for set-variables. Instead we have a quantifier which can say that "most" countable sets have some property. What does "most" mean when we talk about countable sets? Here we use the concept of a club set. A set C of countable subsets of M is *unbounded* if for every countable $X \subseteq M$ there is $Y \in C$ such that $X \subseteq Y$. The set C is *closed* if it is closed under unions of increasing ω-chains. A *club* set means a closed unbounded set. The club (countable) subsets of a set M form a (normal) filter[14], which however is not an ultrafilter. Still, it is a useful measure of largeness. We adopt the following generalized second order quantifier:

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[14] For the definitions of filter and ultrafilter, see [31].

$$\mathfrak{M} \models aas\varphi(s, \vec{a}) \quad \Longleftrightarrow \quad \{A \subseteq M \ : \ |A| \le \omega, (\mathfrak{M}, A) \models \varphi(A, \vec{a})\}$$
$$\text{contains a club of countable subsets of } M.$$

As proved in [50] and [3], the logic $L(aa)$ is countably compact and has a nice complete axiomatization. Of course, it does not have the Löwenheim property, but every consistent sentence has a model of cardinality at most $\aleph_1$. It extends both $L(Q_1)$ and $L(Q_\omega^{\mathrm{cof}})$. The logic $L(aa)$ does not have Beth property [41], but:

▶ **Theorem 27** ([51]). $\mathrm{Craig}(L(Q_\omega^{\mathit{cof}}), L(aa))$.

Proof. (A rough sketch) We actually prove $\mathrm{Rob}(L(Q_\omega^{\mathrm{cof}}), L(aa))$. This involves proving that if T_1 and T_2 are countable complete $L(aa)$-theories, $T_0 = T_1 \cap T_2$ is complete, and $\tau(T_1) \cap \tau(T_2) = \tau(T_0)$, then $(T_1 \cap L(Q_\omega^{\mathrm{cof}})) \cup (T_2 \cap L(Q_\omega^{\mathrm{cof}}))$ is consistent. Theories T_1 and T_2 are first enriched by new unary predicates P_α, $\alpha < \omega_1$. We add the axiom $\varphi(P_{\alpha_1}, \dots, P_{\alpha_n})$ to T_l', whenever $\alpha_1 < \dots < \alpha_n < \omega_1$ and $aas_1 \dots aas_n \varphi(s_1, \dots, s_n) \in T_l$. These predicates build a canonical club in our final model giving us control of $L(aa)$-truth. To see how this works in a simple case, consider the set of

$$\forall x_1 \dots \forall x_n ((P(x_1) \wedge \dots \wedge P(x_{n+1})) \to \varphi^{(P)}(x_1, \dots, x_n)), \tag{2}$$

where

$$aas \forall x_1 \dots \forall x_n ((s(x_1) \wedge \dots \wedge s(x_{n+1})) \to \varphi^{(s)}(x_1, \dots, x_n)) \in T_1 \tag{3}$$

and $\varphi(x_1, \dots, x_n)$ is first order. Since every model has a closed unbounded set of countable subsets that are domains of elementary submodels with respect to first order logic,[15] and the club sets form a filter, the collection of all sentences (2) essentially "says" in a model $\mathfrak{M}$ that $P^\mathfrak{M}$ is the domain of a countable elementary submodel of $\mathfrak{M}$.

Next T_1' and T_2' are reduced to complete first order theories T_1^*, T_2^* by introducing new relation symbols, namely an n-ary symbol R_φ for each formula $\varphi(x_1, \dots, x_n)$ of $L(aa)$, with the defining axiom

$$\forall x_1 \dots x_n (R_\varphi(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n)),$$

and taking only the resulting first order formulas.

Then we use the Robinson property of first order logic to obtain a model $\mathfrak{M}$ of $T_1^* \cup T_2^*$. W.l.o.g. $\mathfrak{M}$ is $\aleph_2$-saturated (see e.g. [29, p. 480] for the definition and basic properties of $\aleph_2$-saturation). Let $\mathfrak{N}$ be the union of the submodels of $\mathfrak{M}$ determined by the predicates P_α. Now we can show that if $\mathfrak{N}$ thinks the cofinality of a definable linear order is uncountable, it is in $\mathfrak{N}$ exactly $\aleph_1$. On the other hand, if $\mathfrak{N}$ thinks the cofinality of a definable linear order is ω,

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[15] One first forms Skolem-functions and then takes submodels that are closed under the Skolem-functions. The collections of such submodels is always closed unbounded.

the real cofinality of the linear order in $\mathfrak{N}$ is actually $\aleph_2$. The final step is a chain argument, using a chain of length ω, to turn cofinality $\aleph_2$ to cofinality $\aleph_0$ without losing the cofinalities that are $\aleph_1$.   ◄

Why the above result is remarkable is that it is the closest we have got so far in obtaining interpolation among countably compact proper extensions of $L_{\omega\omega}$. The situation raises the following intriguing question:

Problem: Is there a logic L such that $L(Q_\omega^{\mathrm{cof}}) \leq L \leq L(aa)$ and L has the Craig interpolation property?

Note that if a logic L as in the above problem exists, then L is also countably compact.

In full single-sorted second order logic L^2, with variables for arbitrary relations, interpolation (even uniform[16]) holds trivially, because we can quantify over relations. The separation property, in particular, becomes trivially true as single-sorted $PC(L^2)$-classes are always $EC(L^2)$-classes. In particular, $\Delta_1^1(L^2) \equiv L^2$. In many-sorted second order logic interpolation (even the weak Beth property) fails because we can use new sorts to define truth implicitly, but not explicitly ([14], [27]), imitating the proof of Theorem 20. The difference between the single-sorted and the many-sorted case is that in the latter the predicates that one would like to quantify away may involve new sorts, i.e. elements outside the current model. There is no way L^2 can reach out to them. In particular, $\Delta(L^2) \not\equiv L^2$.

Problem: Is there any (set-theoretically definable) extension of second order logic that has many-sorted interpolation?

If we drop the requirement of set-theoretical definability, there is a solution: sort logic ([57]). If we just want a proper extension of first order logic with many-sorted interpolation, the logic $L_{\omega_1\omega}$ offers a solution, as we have seen.

Existential second order logic ESO (i.e. single-sorted $PC(L_{\omega\omega})$) is a perfectly nice abstract logic, only it is not closed under negation. It satisfies (even uniform) single-sorted Craig Interpolation Theorem, as we can quantify over relations, i.e. in the single-sorted context $PC(\mathrm{ESO}) \equiv \mathrm{ESO}$. For the same reason ESO satisfies the Souslin-Kleene Interpolation and Beth Theorems. In each case the claim is trivially true. The usual proof of the Robinson property works also for ESO. The logic ESO satisfies also compactness and has the Löwenheim property. This does not violate Lindström's Theorem, because ESO is not closed under negation. In [54] it is proved that there is no strongest abstract logic without negation which is compact and satisfies the Löwenheim property. Many-sorted $PC(L_{\omega\omega})$ is actually equivalent to single-sorted $PC(L_{\omega\omega})$ ([38]). Thus also many-sorted ESO satisfies interpolation and the related properties following from interpolation.

Dependence logic $\mathcal{D}$, based on the atom $=(\vec{x}, \vec{y})$, which says that $\vec{x}$ totally determines $\vec{y}$, was introduced in [55]. On the level of sentences $\mathcal{D}$ is equivalent to ESO, hence it satisfies

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[16] See [9]

(even uniform) Craig Interpolation, Souslin-Kleene Interpolation, Beth theorem, and the Robinson property, whether single- or many-sorted. The Craig Interpolation property holds for $\mathcal{D}$ formulas also in the following form: Suppose $\varphi(\vec{x}, \vec{y}) \models \psi(\vec{x}, \vec{z})$, where $\vec{y} \cap \vec{z} = \emptyset$. Then there is $\theta(\vec{x})$ such that $\varphi(\vec{x}, \vec{y}) \models \theta(\vec{x})$ and $\theta(\vec{x}) \models \psi(\vec{x}, \vec{z})$. Namely, we can take $\theta(\vec{x})$ to be $\exists \vec{y} \varphi(\vec{x}, \vec{y})$. This works, because of locality[17] ([55, Lemma 3.27]).

Inclusion Logic $\mathcal{I}$ ([24]), based on atoms of the form $\vec{x} \subseteq \vec{y}$, which say that every value of $\vec{x}$ occurs as a value of $\vec{y}$, is not equal to ESO on the level of sentences. Rather, inclusion logic is equivalent to Positive Greatest Fixed Point Logic GFP$^+$, i.e. the fragment of Greatest Fixed Point Logic in which fixed point operators occur only positively ([25]). Thus we cannot use reduction to ESO to infer any interpolation properties for inclusion logic.

Problem: Does Inclusion Logic $\mathcal{I}$ ([24]) satisfy the Craig Interpolation property?

6   Conclusion

Interpolation in all its forms mostly fails in extensions of first order logic. It seems difficult to extend first order logic in a way which leads to the kind of balance required by interpolation and the Beth property. There is the artificial way of using the Δ-operation to obtain $\Delta(L)$, which always has the Souslin-Kleene Interpolation property and therefore the weak Beth property. But there is no general method to find a syntax for the semantically defined $\Delta(L)$. The infinitary logic $L_{\omega_1 \omega}$ is remarkable in the richness of its model theory and, as we have seen, it has the interpolation property. In bigger infinitary logics interpolation systematically fails, signalling, perhaps, that there is something incomplete in their syntax. The new large infinitary logic L^1_κ enjoys interpolation, but lacks so far a satisfactory syntax. Overall, the area abounds with open problems, only some of which have been mentioned above.

Acknowledgments

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[17] Locality of a formula means that the truth of the formula depends only on values of assignments on variables which are free in the formula.

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