Uniform guarded fragments: interpolation and complexity

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- GF has several nice meta-logical properties. For example, it has a (generalized) tree-model property, it is decidable and it has the Łoś–Tarski preservation property.
- It does not, however, have the Craig interpolation property (CIP).

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- **Question:** what are the largest fragment(s) of GF with CIP?

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  is not.
- Contains the two-variable fragment FO^2 of FO. Decidable and its satisfiability problem has the same complexity as FO^2 [Kieronski and Kuusisto, 2014].

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  - The above implies that there exists structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \varphi, \mathfrak{B} \models \neg\psi$ and there is a $\mathrm{UGF}_1[\sigma]$-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$. Here σ is the common vocabulary of φ and ψ.

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Theorem (Jaakkola, 2024)

Let φ be a sentence of $\mathrm{UFG}[\sigma_1]$ and ψ be a sentence of $\mathrm{UFG}[\sigma_2]$. If $\varphi \models \psi$, then there exists a sentence θ of $\mathrm{GF}[\sigma_1 \cap \sigma_2]$ such that $\varphi \models \theta \models \psi$.

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Theorem ([Jaakkola, 2022])

The satisfiability problem for UGF *is* $\mathrm{N}\textsc{ExpTime}$*-complete.*

Thanks!

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