First-order interpolation via polyadic spaces

Sam van Gool
joint work with Jérémie Marquès

IRIF, Université Paris Cité

Workshop on Theory and Applications
of Craig Interpolation and Beth Definability

Amsterdam, 22 April 2024

Logic and interpolation via algebra

In this work, we take an *algebraic* perspective on logic:

- The *free Heyting algebra* on a set X is the set of classes of propositional formulas in variables X under the relation

$$\varphi \equiv \psi \iff \text{the formula } \varphi \leftrightarrow \psi \text{ is an intuitionistic tautology.}$$

Logic and interpolation via algebra

In this work, we take an *algebraic* perspective on logic:

- The *free Heyting algebra* on a set X is the set of classes of propositional formulas in variables X under the relation

  $\varphi \equiv \psi \iff$ the formula $\varphi \leftrightarrow \psi$ is an intuitionistic tautology.

- Similar algebras exist for other *propositional* logics, and can be used to establish *interpolation* theorems.

Logic and interpolation via algebra

In this work, we take an *algebraic* perspective on logic:

- The *free Heyting algebra* on a set X is the set of classes of propositional formulas in variables X under the relation

  $$\varphi \equiv \psi \iff \text{the formula } \varphi \leftrightarrow \psi \text{ is an intuitionistic tautology.}$$

- Similar algebras exist for other *propositional* logics, and can be used to establish *interpolation* theorems.

- **Question.** What about *first-order* logics?

Logic and interpolation via algebra

In this work, we take an *algebraic* perspective on logic:

- The *free Heyting algebra* on a set X is the set of classes of propositional formulas in variables X under the relation

$$\varphi \equiv \psi \iff \text{the formula } \varphi \leftrightarrow \psi \text{ is an intuitionistic tautology.}$$

- Similar algebras exist for other *propositional* logics, and can be used to establish *interpolation* theorems.

- **Question.** What about *first-order* logics?

- Here, we look at one particular tool, called *polyadic spaces*, which may be used for this.

Craig interpolation algebraically

The Craig interpolation property of intuitionistic propositional logic is equivalent to:

Theorem (Pitts 1983, cf. also Maksimova 1977)

For any sets X and Y, the following pushout square of free Heyting algebras has the interpolation property:

$$\begin{array}{ccc} F(X \cap Y) & \xhookrightarrow{\ f\ } & F(Y) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ F(X) & \xhookrightarrow{\ v\ } & F(X \cup Y) \end{array}$$

Craig interpolation algebraically

The Craig interpolation property of intuitionistic propositional logic is equivalent to:

Theorem (Pitts 1983, cf. also Maksimova 1977)

For any sets X and Y, the following pushout square of free Heyting algebras has the interpolation property:

$$\begin{array}{ccc} F(X \cap Y) & \xhookrightarrow{\quad f \quad} & F(Y) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ F(X) & \xhookrightarrow{\quad v \quad} & F(X \cup Y) \end{array}$$

Here, the *interpolation property* for such a square says: whenever $u(\varphi) \leq v(\psi)$, there exists θ such that $\varphi \leq f(\theta)$ and $g(\theta) \leq \psi$.

An extension to predicate logic

Pitts also showed that:

- *Any* pushout square of Heyting algebras has the above interpolation property.

An extension to predicate logic

Pitts also showed that:

- *Any* pushout square of Heyting algebras has the above interpolation property.

- The same works for *Heyting pretoposes*, which yields a proof of Craig interpolation for intuitionistic *predicate* logic, as well.

An extension to predicate logic

Pitts also showed that:

- *Any* pushout square of Heyting algebras has the above interpolation property.
- The same works for *Heyting pretoposes*, which yields a proof of Craig interpolation for intuitionistic *predicate* logic, as well.
- We revisit Pitts' proof and the interpolation property from the perspective of Joyal's *polyadic spaces*.

Intuitionistic theories via hyperdoctrines

Let T be an intuitionistic first-order theory.

Define a functor $\mathcal{D}_T \colon \mathbf{FinSet} \to \mathbf{HeytAlg}$ by

$$\mathcal{D}_T(n) := \{\varphi \mid \text{free variables of } \varphi \text{ are in } n\}/{\equiv_T}$$

for n any finite set, where $\varphi \equiv_T \psi$ means $T \vdash \varphi \leftrightarrow \psi$, and

Intuitionistic theories via hyperdoctrines

Let T be an intuitionistic first-order theory.

Define a functor $\mathcal{D}_T \colon \mathbf{FinSet} \to \mathbf{HeytAlg}$ by

$$\mathcal{D}_T(n) := \{\varphi \mid \text{free variables of } \varphi \text{ are in } n\}/{\equiv_T}$$

for n any finite set, where $\varphi \equiv_T \psi$ means $T \vdash \varphi \leftrightarrow \psi$, and

$$\mathcal{D}_T(f) \ : \ \varphi(x_1, \ldots, x_n) \mapsto \varphi(f(x_1), \ldots, f(x_m)) \ ,$$

for any function $f \colon n \to m$ between finite sets.

Intuitionistic theories via hyperdoctrines

Let T be an intuitionistic first-order theory.

Define a functor $\mathcal{D}_T\colon \mathbf{FinSet} \to \mathbf{HeytAlg}$ by

$$\mathcal{D}_T(n) := \{\varphi \mid \text{free variables of } \varphi \text{ are in } n\}/{\equiv_T}$$

for n any finite set, where $\varphi \equiv_T \psi$ means $T \vdash \varphi \leftrightarrow \psi$, and

$$\mathcal{D}_T(f) \;:\; \varphi(x_1, \ldots, x_n) \mapsto \varphi(f(x_1), \ldots, f(x_m)) \;,$$

for any function $f\colon n \to m$ between finite sets.

The notion of an *intuitionistic hyperdoctrine* axiomatizes the functors arising in this way (variants: *coherent*, *Boolean*, ...).

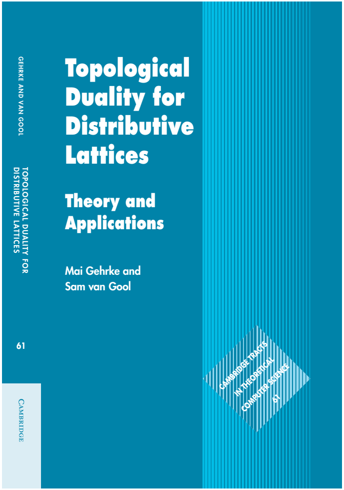
Duality for Heyting algebras and openness

The category of Heyting algebras is dually equivalent to the category of *Esakia spaces*, i.e., compact totally order disconnected spaces such that $\downarrow U$ is open for any open subset U.

Duality for Heyting algebras and openness

The category of Heyting algebras is dually equivalent to the category of *Esakia spaces*, i.e., compact totally order disconnected spaces such that $\downarrow U$ is open for any open subset U.

...



Gehrke & SvG, *Topological duality for distributive lattices: Theory and Applications*. Cambridge University Press (2024).

Duality for Heyting algebras and openness

The category of Heyting algebras is dually equivalent to the category of *Esakia spaces*, i.e., compact totally order disconnected spaces such that $\downarrow U$ is open for any open subset U.

We call a function g between compact ordered spaces:

- *lower semi-open* if $\uparrow g[U]$ is open for any open up-set U,
- *upper semi-open* if $\downarrow g[U]$ is open for any open down-set U.

When g is the dual of a Heyting homomorphism h, these conditions correspond to h having a Frobenius left or right adjoint, respectively.

Duality for intuitionistic hyperdoctrines

Theorem

The category of intuitionistic hyperdoctrines on $\mathbf{C}$ *is dually equivalent to the category of functors* $\mathcal{S} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Esakia}$ *such that:*

1. $\mathcal{S}$ *sends pushout squares to interpolation squares;*
2. *for any morphism* f *of* $\mathbf{C}$, $\mathcal{S}(f)$ *is lower and upper semi-open.*

These functors are called *intuitionistic polyadic spaces*, and a large part of the theory is due to Joyal (1971 and 2019).

Interpolation via polyadic spaces

For intuitionistic propositional logic, one may prove Craig interpolation by taking a *fibered product* of Kripke models.

With polyadic spaces, one follows a similar idea, after defining the appropriate notions of *model* and *product* for polyadic spaces.

The proof resembles that of Pitts, but is done dually, and extends to the more general setting of *compact ordered spaces*.

See: SvG & Marquès (2024), Section 11; Marquès (2023), Section 3.3.

An open problem

Mints, Olkhovikov, Urquhart (2013): the ‘constant domain’ intuitionstic predicate logic **CD**, given by the axiom scheme

$$\forall x(\varphi(x) \vee \psi) \leftrightarrow (\forall x \varphi(x)) \vee \psi \quad \text{(CD)}$$

does *not* have interpolation.

An open problem

Mints, Olkhovikov, Urquhart (2013): the 'constant domain' intuitionstic predicate logic **CD**, given by the axiom scheme

$$\forall x(\varphi(x) \vee \psi) \leftrightarrow (\forall x \varphi(x)) \vee \psi \quad \text{(CD)}$$

does *not* have interpolation.

Open problem.

Does Gödel logic (**G** = **CD** + Linearity) have interpolation?

$$(\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{(Linearity)}$$

An open problem

Mints, Olkhovikov, Urquhart (2013): the 'constant domain' intuitionstic predicate logic **CD**, given by the axiom scheme

$$\forall x(\varphi(x) \vee \psi) \leftrightarrow (\forall x \varphi(x)) \vee \psi \quad \text{(CD)}$$

does *not* have interpolation.

Open problem.

Does Gödel logic ($\mathbf{G} = \mathbf{CD}$ + Linearity) have interpolation?

$$(\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{(Linearity)}$$

Our hope was (is?) to use the above machinery to solve it.

Our note Baaz, Gehrke, SvG (2018) shows that the counterexample used for **CD** *does* have an interpolant in **G**.

An open problem

Mints, Olkhovikov, Urquhart (2013): the 'constant domain' intuitionstic predicate logic **CD**, given by the axiom scheme

$$\forall x(\varphi(x) \vee \psi) \leftrightarrow (\forall x \varphi(x)) \vee \psi \quad \text{(CD)}$$

does *not* have interpolation.

Open problem.

Does Gödel logic ($\mathbf{G} = \mathbf{CD}$ + Linearity) have interpolation?

$$(\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{(Linearity)}$$

Our hope was (is?) to use the above machinery to solve it.

Our note Baaz, Gehrke, SvG (2018) shows that the counterexample used for **CD** *does* have an interpolant in **G**.

There is work to do!