On Craig Interpolation in SMT

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2024-04-22
CIBD Workshop, Amsterdam

Outline

- Craig interpolation in verification
- Summary of some interpolation results for theories
- SMT solvers supporting Craig interpolation
- Beyond binary interpolation

Motivation: inference of invariants

Generic verification problem ("safety")

$$\{ \texttt{pre} \} \texttt{ while (*) Body } \{ \texttt{post} \}$$

Standard approach: loop rule using invariant

$$\frac{\texttt{pre} \Rightarrow \phi \qquad \{ \phi \} \texttt{ Body } \{ \phi \} \qquad \phi \Rightarrow \texttt{post}}{\{ \texttt{pre} \} \texttt{ while (*) Body } \{ \texttt{post} \}}$$

How to compute ϕ automatically?

From intermediate assertions to invariants

$\{\texttt{pre}\}$ Body; Body $\{\texttt{post}\}$?

↓

Bounded model checking problem ✓

↓

Compute intermediate assertion ψ_1

↓

$\{\texttt{pre}\}$ Body $\{\psi_1\}$ $\{\psi_1\}$ Body $\{\texttt{post}\}$

[McMillan, 2003]

From intermediate assertions to invariants

$\{\texttt{pre}\}$ Body; Body $\{\texttt{post}\}$?

Bounded model checking problem ✓

Compute intermediate assertion ψ_1

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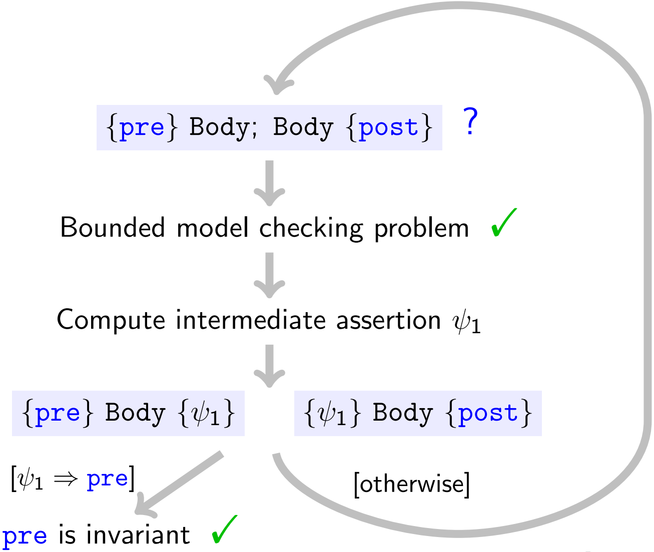
$[\psi_1 \Rightarrow \texttt{pre}]$

pre is invariant ✓

[McMillan, 2003]

From intermediate assertions to invariants



[McMillan, 2003]

From intermediate assertions to invariants

$\{\texttt{pre} \vee \psi_1\}\ \texttt{Body};\ \texttt{Body}\ \{\texttt{post}\}$?

Bounded model checking problem ✓

Compute intermediate assertion ψ_2

$\{\texttt{pre} \vee \psi_1\}\ \texttt{Body}\ \{\psi_2\}$ $\{\psi_2\}\ \texttt{Body}\ \{\texttt{post}\}$

$[\psi_1 \Rightarrow \texttt{pre}]$ [otherwise]

$\texttt{pre}$ is invariant ✓

[McMillan, 2003]

From intermediate assertions to invariants

$\{\texttt{pre} \lor \psi_1\}$ Body; Body $\{\texttt{post}\}$?

Bounded model checking problem ✓

Compute intermediate assertion ψ_2

$\{\texttt{pre} \lor \psi_1\}$ Body $\{\psi_2\}$ $\{\psi_2\}$ Body $\{\texttt{post}\}$

$[\psi_2 \Rightarrow \texttt{pre} \lor \psi_1]$

$\texttt{pre} \lor \psi_1$ is invariant ✓

[otherwise]

[McMillan, 2003]

From intermediate assertions to invariants

$\{\texttt{pre} \vee \psi_1\}$ Body; Body $\{\texttt{post}\}$?

↓

Bounded model checking problem ✓

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Compute intermediate assertion ψ_2

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$\{\texttt{pre} \vee \psi_1\}$ Body $\{\psi_2\}$ $\{\psi_2\}$ Body $\{\texttt{post}\}$

$[\psi_2 \Rightarrow \texttt{pre} \vee \psi_1]$ ...

$\texttt{pre} \vee \psi_1$ is invariant ✓

[McMillan, 2003]

How to compute intermediate assertions?

VC generation

{ pre }	pre (s_0)
Body;	$\to$ Body (s_0, s_1)
Body	$\to$ Body (s_1, s_2)
{ post }	$\to$ post (s_2)

How to compute intermediate assertions?

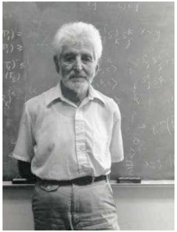
VC generation

{ pre }	$\texttt{pre}(s_0)$
Body;	$\to \texttt{Body}(s_0, s_1)$
Body	$\to \texttt{Body}(s_1, s_2)$
{ post }	$\to \texttt{post}(s_2)$

Theorem (Craig, 1957)

Suppose $A \to C$ is a valid implication. A formula I is called a Craig interpolant if

- *$A \to I$ and $I \to C$ are valid,*
- *every non-logical symbol of I occurs in both A and C.*



How to compute intermediate assertions?

VC generation

{ `pre` }	`pre` (s_0)	$A(s_0, s_1)$
`Body;`	$\rightarrow$ `Body` (s_0, s_1)	$\downarrow$
....	....	$I(s_1)$
`Body`	$\rightarrow$ `Body` (s_1, s_2)	$\downarrow$
{ `post` }	$\rightarrow$ `post` (s_2)	$C(s_1, s_2)$

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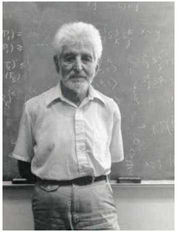

Abstraction with interpolants

{pre} Body; Body {post} ?

↓

Bounded model checking problem ✓

↓

Compute intermediate assertion ψ_1

↓

...

Abstraction with interpolants

{pre} Body; Body {post} ?

↓

Bounded model checking problem ✓

↓

Compute intermediate assertion ψ_1

↓

...

Interpolant extracted from proof
⇒
Abstraction from unnecessary details

Theories

- Following [McMillan 2003], several solvers and theorem provers add interpolation support:
  - SAT solvers
  - Foci $\rightarrow$ iZ3 $\rightarrow$ Z3
  - MathSAT
  - CLPprover
  - CSIsat
  - OpenSMT
  - Princess
  - SMTInterpol
  - Vampire
  - AXDInterpolator
  - *etc.*
- "Race" to find interpolation procedures for relevant theories.

Standard SMT theories

- EUF
- Arrays
- LRA
- LIA
- NRA
- NIA
- Bit-vectors
- Floats
- ADTs
- Strings
- *(+ combinations)*

Towards Satisfiability Modulo Theories paradigm (SMT)

- Satisfiability Modulo Theories (SMT) solvers are today the standard backends in verification
- Maintained solvers supporting Craig interpolation:

Solver	...
MathSAT5	
OpenSMT2	
Princess	
SMTInterpol	
cvc5	
Vampire	
Z3	

- *(any tools missing?)*

Reverse interpolants

- It is common in verification to use the following variant of interpolation:

Definition

Suppose $A \wedge B$ is unsatisfiable. A *reverse interpolant* is a formula I such that

- $A \rightarrow I$ and $B \rightarrow \neg I$ are valid,
- every non-logical symbol of I occurs in both A and B.

Lemma

In classical logic, reverse interpolants and ordinary interpolants are interchangeable:

$$I \text{ is reverse interpolant for } A \wedge B \quad \Longleftrightarrow \quad I \text{ is interpolant for } A \rightarrow \neg B$$

Interpolation in theories

Theorem (Kovacs, Voronkov, 2009)

Suppose T is a theory and $A \wedge B$ a T-unsatisfiable conjunction in first-order logic:

$$A \wedge B \models_T \mathit{false}$$

Then there is a formula I such that:

- $A \models_T I$
- $B \models \neg I$
- *every non-logical symbol ...*

- *Problem:* even if $A \wedge B$ is quantifier-free, the I might contain quantifiers.
- Often a problem in verification.

Plain quantifier-free theory interpolation

Definition (Bruttomesso, Ghilardi, Ranise, 2014)

A theory T admits *plain quantifier-free interpolation* if for every quantifier-free T-unsatisfiable conjunction $A \wedge B$ (with arbitrary free variables, but otherwise only containing T-symbols) there is a quantifier-free formula I with:

- $A \models_T I$
- $B \models_T \neg I$
- every variable in I occurs in both A and B.

General quantifier-free theory interpolation

Definition (Bruttomesso, Ghilardi, Ranise, 2014)

A theory T admits *general quantifier-free interpolation* if for every closed quantifier-free T-unsatisfiable conjunction $A \wedge B$ (with symbols from T, but also including other functions or predicates) there is a quantifier-free (reverse) interpolant I.

- Plain and general QFI can be characterized in terms of (sub-)amalgamation.
- The second property is equivalent to the notion of *equality interpolation*, and important for theory combination.

The Big Picture

	EUF	Arrays	LRA	LIA	NRA	NIA	BV	Floats	ADT	Strings
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gen. QFI										

The Big Picture

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gen. QFI	✓		✓							

- Kenneth L. McMillan: An interpolating theorem prover. Theor. Comput. Sci. 345(1): 101-121 (2005)

Interpolating LRA

LRA proof rules

$$\frac{s \geq 0 \qquad t \geq 0}{\alpha s + \beta t \geq 0} \quad (\text{for } \alpha, \beta \geq 0) \qquad\qquad \frac{\alpha \geq 0}{\square} \quad (\text{for } \alpha < 0)$$

Interpolating LRA (2)

Interpolating LRA proof rules

- Annotate every inequality with a *partial interpolant:*

$$\frac{s \geq 0 \text{ is a formula from } A}{s \geq 0 \; [s]} \qquad \frac{s \geq 0 \text{ is a formula from } B}{s \geq 0 \; [0]}$$

- Propagate those partial interpolants:

$$\frac{s \geq 0 \; [s'] \qquad t \geq 0 \; [t']}{\alpha s + \beta t \geq 0 \; [\alpha s' + \beta t']} \quad (\text{for } \alpha, \beta \geq 0) \qquad \frac{\alpha \geq 0 \; [s']}{\Box \; [s' \geq 0]} \quad (\text{for } \alpha < 0)$$

- The partial interpolant annotating $\Box$ is an interpolant for $A \wedge B$.

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- Similar rules can be defined for EUF.

Interpolation paradigms

1. Proof-based
    1.1 Bottom-up: propagate partial interpolants ("resolution-style")

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- Every theory that admits quantifier elimination also has plain quantifier-free interpolation.

[1] Needs a divisibility operator $|$.

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- Interpolants computed using quantifier elimination tend to be less useful in verification: no “abstraction from unnecessary details”

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- Interpolants computed using quantifier elimination tend to be less useful in verification: no “abstraction from unnecessary details”

[1] Needs a divisibility operator $|$.
[2] Needs a *diff* function, see Silvio’s talk.

Proof-based LIA Interpolation

LIA proof rules

LRA proof rules + some combination of:

- Branch & bound:

$$\frac{x = \alpha}{x \leq \lfloor \alpha \rfloor \quad x \geq \lceil \alpha \rceil}$$

- Cuts:

$$\frac{\sum_i \alpha_i x_i + \beta \geq 0}{\sum_i \frac{\alpha_i}{\gamma} x_i + \lfloor \frac{\beta}{\gamma} \rfloor \geq 0} \quad (\gamma > 0 \text{ divides all } \alpha_i)$$

- Strengthening (e.g., Omega test):

$$\frac{t \geq n}{t = n \quad t \geq n+1}$$

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- Splitting requires a further paradigm in interpolation . . .

Interpolation paradigms

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   1.2 **Top-down: recursive computation of interpolants**
2. Graph-based: summarize edges in an e-graph
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Computation of interpolants with splitting

$$\frac{A_1 \vee A_2, B \;\triangleright\; I_1 \vee I_2}{A_1, B \;\triangleright\; I_1 \qquad A_2, B \;\triangleright\; I_2} \qquad\qquad \frac{A, B_1 \vee B_2 \;\triangleright\; I_1 \wedge I_2}{A, B_1 \;\triangleright\; I_1 \qquad A, B_2 \;\triangleright\; I_2}$$

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- For poly-size interpolants: either integer division $\div$ or bounded quantifiers needed.

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- Alberto Griggio, Thi Thieu Hoa Le, Roberto Sebastiani: Efficient Interpolant Generation in Satisfiability Modulo Linear Integer Arithmetic. TACAS 2011: 143-157
- Angelo Brillout, Daniel Kroening, PR, Thomas Wahl: An Interpolating Sequent Calculus for Quantifier-Free Presburger Arithmetic. J. Autom. Reason. 47(4): 341-367 (2011)

[1] Needs ~~a divisibility operator $\mid$~~ integer division $\div$ or bounded quantifiers.
[2] Needs a *diff* function.

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- Peter Backeman, PR, Aleksandar Zeljic: Bit-Vector Interpolation and Quantifier Elimination by Lazy Reduction. FMCAD 2018: 1-10

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2. Graph-based: summarize edges in an e-graph
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4. **Reduction-based: by mapping interpolation problem to another theory**

- Deepak Kapur, Rupak Majumdar, Calogero G. Zarba: Interpolation for data structures. SIGSOFT FSE 2006: 105-116

Fixed-length bit-vectors

- Formalization of machine arithmetic, very widely used in verification
- Domains $x \in \mathbb{B}^n$, often for $n = 32$ or $n = 64$
- Different classes of operations:
  - Arithmetic: $\text{bvadd}, \text{bvmul}, \dots$
  - Sequence: $\text{concat}, \text{extract}, \text{shift}, \dots$
  - Bit-wise: $\text{bvand}, \text{bvor}, \dots$
- Though finite, often resulting in very hard constraints

Bit-vector interpolation by reduction

Approaches

- Approach 1: reduction to propositional logic $\rightarrow$ “bit-blasting”
- Approach 2: reduction to LIA/NIA
- Approach 3: lazy reduction to LIA/NIA

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- Approach 1: reduction to propositional logic $\rightarrow$ "bit-blasting"
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  - Low-level propositional interpolants, less useful for software verification
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- Approach 2: reduction to LIA/NIA
  - Good for formulas with mostly linear, arithmetic operations
  - Due to overflows, often leads to hard LIA formulas and convoluted interpolants
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- A. Griggio, "Effective word-level interpolation for software verification," FMCAD 2011

Bit-vector interpolation by reduction

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- Approach 1: reduction to propositional logic $\rightarrow$ "bit-blasting"
  - Good for formulas with many bit-wise operations
  - Low-level propositional interpolants, less useful for software verification
- Approach 2: reduction to LIA/NIA
  - Good for formulas with mostly linear, arithmetic operations
  - Due to overflows, often leads to hard LIA formulas and convoluted interpolants
- Approach 3: lazy reduction to LIA/NIA
  - Good for formulas with mostly arithmetic operations; much "nicer" interpolants
  - Still difficult to support bit-wise operations efficiently

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- Hossein Hojjat, PR: Deciding and Interpolating Algebraic Data Types by Reduction. SYNASC 2017: 145-152

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3. Quantifier elimination
4. Reduction-based: by mapping interpolation problem to another theory
5. **Constraint-based: systematic search for interpolants in some language**
   - Syntax-guided synthesis
   - Linear arithmetic constraint solving

Interpolation support in SMT solvers *(apologies for errors!)*

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Z3									

Interpolation support in SMT solvers *(apologies for errors!)*

	EUF	Arrays	LRA	LIA	NRA	NIA	BV	Floats	ADT
plain QFI	✓	✓	✓	✓	✓	✗	✓	✓	✓
gen. QFI	✓	✓	✓	✓			✓	✓	✓
MathSAT5	✓	?	✓	✓			✓	?	
OpenSMT2	✓		✓	✓					
Princess	✓	✓		✓		✓	✓		✓
SMTInterpol	✓	✓	✓	✓					
cvc5									
Vampire									
Z3									

Interpolation support in SMT solvers *(apologies for errors!)*

	EUF	Arrays	LRA	LIA	NRA	NIA	BV	Floats	ADT
plain QFI	✓	✓	✓	✓	✓	✗	✓	✓	✓
gen. QFI	✓	✓	✓	✓			✓	✓	✓
MathSAT5	✓	?	✓	✓			✓	?	
OpenSMT2	✓		✓	✓					
Princess	✓	✓		✓		✓	✓		✓
SMTInterpol	✓	✓	✓	✓					
cvc5	✓[1]	✓	✓	✓	✓	✓	✓	✓	✓
Vampire									
Z3									

[1]Via syntax-guided synthesis.

Interpolation support in SMT solvers *(apologies for errors!)*

	EUF	Arrays	LRA	LIA	NRA	NIA	BV	Floats	ADT
plain QFI	✓	✓	✓	✓	✓	✗	✓	✓	✓
gen. QFI	✓	✓	✓	✓			✓	✓	✓
MathSAT5	✓	?	✓	✓			✓	?	
OpenSMT2	✓		✓	✓					
Princess	✓	✓		✓		✓	✓		✓
SMTInterpol	✓	✓	✓	✓					
cvc5	✓[1]	✓	✓	✓	✓	✓	✓	✓	✓
Vampire	✓	✓[2]	✓	✓					
Z3									

[1] Via syntax-guided synthesis.

[2] Focussing on first-order interpolants.

Interpolation support in SMT solvers *(apologies for errors!)*

	EUF	Arrays	LRA	LIA	NRA	NIA	BV	Floats	ADT
plain QFI	✓	✓	✓	✓	✓	✗	✓	✓	✓
gen. QFI	✓	✓	✓	✓			✓	✓	✓
MathSAT5	✓	?	✓	✓			✓	?	
OpenSMT2	✓		✓	✓					
Princess	✓	✓		✓		✓	✓		✓
SMTInterpol	✓	✓	✓	✓					
cvc5	✓[1]	✓	✓	✓	✓	✓	✓	✓	✓
Vampire	✓	✓[2]	✓	✓					
Z3		✓[3]	✓	✓	✓	✓	✓		✓

[1] Via syntax-guided synthesis.
[2] Focussing on first-order interpolants.
[3] Via its constrained Horn clause engine.

Beyond binary interpolation

Extended versions of interpolation

- Sequence interpolants
- Tree interpolants
- Disjunctive interpolants
- DAG interpolants

- All those notions can be reduced to binary/standard interpolation.
- But they are quite widely used: most solvers support sequence and/or tree interpolants.

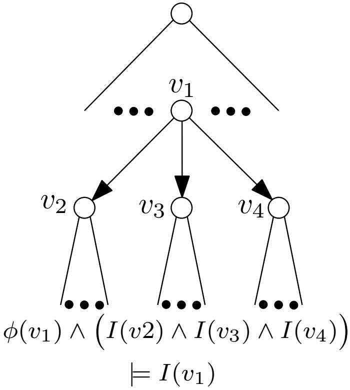
Example: tree interpolation

Tree interpolant

Suppose $T = (V, E)$ is a finite directed tree, and $\phi : V \to For$ a labeling function such that $\bigwedge_{v \in V} \phi(v)$ is unsatisfiable.

$I : V \to For$ is a *tree interpolant* if

- $I(root) = false$
- For all $v \in V$:
  $\phi(v) \wedge \bigwedge_{(v,w) \in E} I(w) \models I(v)$
- Non-logical symbols in $I(v)$ occur both in the sub-tree underneath v and in the rest of the tree.



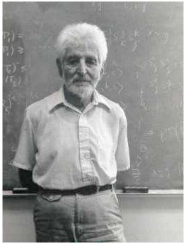
Craig interpolation as recursion-free Horn solving

Observation

- Let A, B be formulas with common variables $\bar{x}$.
- Then:

$$I(\bar{x}) \text{ is a reverse interpolant of } A \wedge B$$
$$\Leftrightarrow$$
$$\text{Formulas } A \rightarrow I(\bar{x}) \text{ and } B \wedge I(\bar{x}) \rightarrow \mathit{false} \text{ are valid}$$

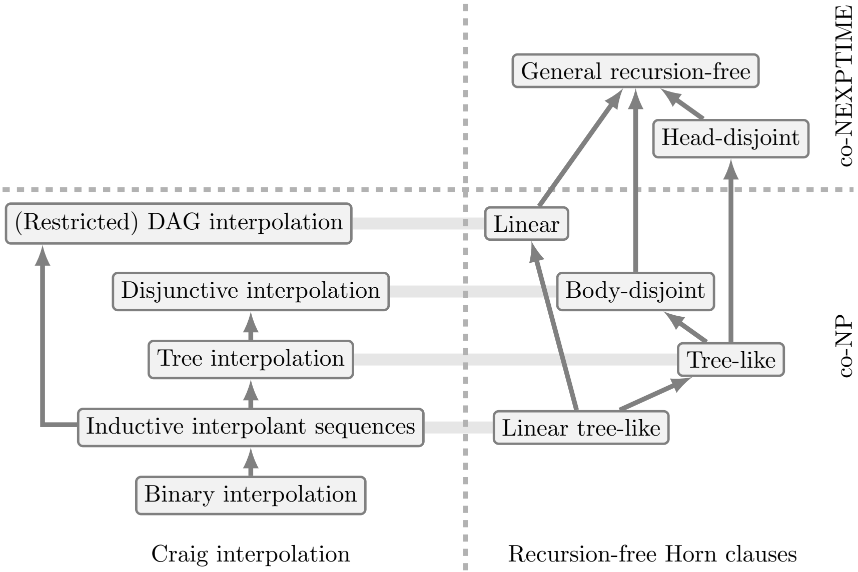
- $A \rightarrow I(\bar{x})$, $B \wedge I(\bar{x}) \rightarrow \mathit{false}$ can be seen as *constrained Horn clauses* over a relation symbol I.
- Correspondence between (extended) Craig interpolants and solution sets of recursion-free Horn clauses



William Craig

Alfred Horn

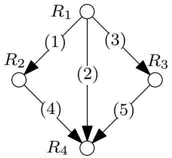
Taxonomy of Recursion-free Horn Clauses & Interpolation



Recursion-Free Horn Clause Fragments

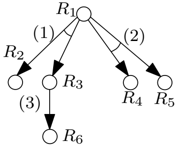
Linear: the body of each clause contains at most one relation symbol.

$$
\begin{aligned}
&1)\, C_1 \wedge R_2(\bar{x}) \rightarrow R_1(\bar{x})\\
&2)\, C_2 \wedge R_4(\bar{x}) \rightarrow R_1(\bar{x})\\
&3)\, C_3 \wedge R_3(\bar{x}) \rightarrow R_1(\bar{x})\\
&4)\, C_4 \wedge R_4(\bar{x}) \rightarrow R_2(\bar{x})\\
&5)\, C_5 \wedge R_4(\bar{x}) \rightarrow R_3(\bar{x})
\end{aligned}
$$



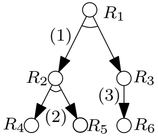
Body-disjoint: each relation symbol occurs at most once in body of a clause.

$$
\begin{aligned}
&1)\, C_1 \wedge R_2(\bar{x}) \wedge R_3(\bar{x}) \rightarrow R_1(\bar{x})\\
&2)\, C_2 \wedge R_4(\bar{x}) \wedge R_5(\bar{x}) \rightarrow R_1(\bar{x})\\
&3)\, C_3 \wedge R_6(\bar{x}) \rightarrow R_3(\bar{x})
\end{aligned}
$$



Tree-like: body-disjoint & head-disjoint: each relation symbol occurs at most once in head of a clause.

$$
\begin{aligned}
&1)\, C_1 \wedge R_2(\bar{x}) \wedge R_3(\bar{x}) \rightarrow R_1(\bar{x})\\
&2)\, C_2 \wedge R_4(\bar{x}) \wedge R_5(\bar{x}) \rightarrow R_2(\bar{x})\\
&3)\, C_3 \wedge R_6(\bar{x}) \rightarrow R_3(\bar{x})
\end{aligned}
$$



Horn solving in verification

- Constrained Horn clauses are considered a "unifying framework" in software model checking
- Horn solvers often internally use Craig interpolation
- Vice versa, Horn solvers are able to compute Craig interpolants

- PR, Hossein Hojjat, Viktor Kuncak: The Relationship between Craig Interpolation and Recursion-Free Horn Clauses. CoRR abs/1302.4187 (2013)

Conclusions

- Consider the talk as the starting point of a systematic survey
- Several dimensions remain to be explored:
  - Support for theory combination
  - Interpolation vs. uniform interpolation
  - Support for quantifiers
  - Complexity
- Comments, questions?

Challenges

- Interpolation for some of the theories:
  - Bit-vectors
  - Floating-point numbers
  - Strings, sequences

- What is a good interpolant? How to search for interpolants?